

BIRATIONAL CLASSIFICATION OF FIELDS OF INVARIANTS FOR GROUPS OF ORDER 128

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ABSTRACT. Let G be a finite group acting on the rational function field $\mathbb{C}(x_g : g \in G)$ by \mathbb{C} -automorphisms $h(x_g) = x_{hg}$ for any $g, h \in G$. Noether's problem asks whether the invariant field $\mathbb{C}(G) = k(x_g : g \in G)^G$ is rational (i.e. purely transcendental) over \mathbb{C} . By Fischer's theorem, $\mathbb{C}(G)$ is rational over \mathbb{C} when G is a finite abelian group. Saltman and Bogomolov, respectively, showed that for any prime p there exist groups G of order p^9 and of order p^6 such that $\mathbb{C}(G)$ is not rational over \mathbb{C} by showing the non-vanishing of the unramified Brauer group: $\text{Br}_{\text{nr}}(\mathbb{C}(G)) \neq 0$, which is an avatar of the birational invariant $H^3(X, \mathbb{Z})_{\text{tors}}$ given by Artin and Mumford where X is a smooth projective complex variety whose function field is $\mathbb{C}(G)$. For $p = 2$, Chu, Hu, Kang and Prokhorov proved that if G is a 2-group of order ≤ 32 , then $\mathbb{C}(G)$ is rational over \mathbb{C} . Chu, Hu, Kang and Kunyavskii showed that if G is of order 64, then $\mathbb{C}(G)$ is rational over \mathbb{C} except for the groups G belonging to the two isoclinism families Φ_{13} with $\text{Br}_{\text{nr}}(\mathbb{C}(G)) = 0$ and Φ_{16} with $\text{Br}_{\text{nr}}(\mathbb{C}(G)) \simeq C_2$. Bogomolov and Böhning's theorem claims that if G_1 and G_2 belong to the same isoclinism family, then $\mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably \mathbb{C} -isomorphic. We investigate the birational classification of $\mathbb{C}(G)$ for groups G of order 128 with $\text{Br}_{\text{nr}}(\mathbb{C}(G)) \neq 0$. Moravec showed that there exist exactly 220 groups G of order 128 with $\text{Br}_{\text{nr}}(\mathbb{C}(G)) \neq 0$ forming 11 isoclinism families Φ_j . We show that if G_1 and G_2 belong to Φ_{16} , Φ_{31} , Φ_{37} , Φ_{39} , Φ_{43} , Φ_{58} , Φ_{60} or Φ_{80} (resp. Φ_{106} or Φ_{114}), then $\mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably \mathbb{C} -isomorphic with $\text{Br}_{\text{nr}}(\mathbb{C}(G_i)) \simeq C_2$. Explicit structures of non-rational fields $\mathbb{C}(G)$ are given for each cases including also the case Φ_{30} with $\text{Br}_{\text{nr}}(\mathbb{C}(G)) \simeq C_2 \times C_2$.

1. INTRODUCTION

Let k be a field and G be a finite group acting on the rational function field $k(x_g : g \in G)$ by k -automorphisms $h(x_g) = x_{hg}$ for any $g, h \in G$. We denote the fixed field $k(x_g : g \in G)^G$ by $k(G)$. Emmy Noether [Noe13, Noe17] asked whether $k(G)$ is rational (= purely transcendental) over k . This is called Noether's problem for G over k , and is related to the inverse Galois problem, to the existence of generic G -Galois extensions over k , and to the existence of versal G -torsors over k -rational field extensions (see Saltman [Sal82a], Swan [Swa83], Manin and Tsfasman [MT86], Garibaldi, Merkurjev and Serre [GMS03, Section 33.1, page 86]), Colliot-Thélène and Sansuc [CTS07]).

Theorem 1.1 (Fischer [Fis15], see also Swan [Swa83, Theorem 6.1]). *Let G be a finite abelian group with exponent e . Assume that (i) either $\text{char } k = 0$ or $\text{char } k = p$ with $p \nmid e$, and (ii) k contains a primitive e -th root of unity. Then $k(G)$ is k -rational. In particular, $\mathbb{C}(G)$ is \mathbb{C} -rational.*

Theorem 1.2 (Kuniyoshi [Kun54], [Kun55], [Kun56], see also Gaschütz [Gas59]). *Let k be a field with $\text{char } k = p > 0$ and G be a finite p -group. Then $k(G)$ is k -rational.*

We now recall some relevant definitions of k -rationality of fields.

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Definition 1.3. Let K/k and L/k be finite generated extensions of fields.

- (1) K is said to be *rational* over k (for short, *k-rational*) if K is purely transcendental over k , i.e. $K \simeq k(x_1, \dots, x_n)$ for some algebraically independent elements x_1, \dots, x_n over k ;
- (2) K is said to be *stably k-rational* if $K(y_1, \dots, y_m)$ is *k-rational* for some algebraically independent elements y_1, \dots, y_m over K ;
- (3) K and L are said to be *stably k-isomorphic* if $K(y_1, \dots, y_m) \simeq L(z_1, \dots, z_n)$ for some algebraically independent elements y_1, \dots, y_m over K and z_1, \dots, z_n over L ;
- (4) (Saltman, [Sal84b, Definition 3.1]) K is said to be *retract k-rational* if there exists a *k*-algebra A contained in K such that (i) K is the quotient field of A , (ii) there exist a non-zero polynomial $f \in k[x_1, \dots, x_n]$ and *k*-algebra homomorphisms $\varphi: A \rightarrow k[x_1, \dots, x_n][1/f]$ and $\psi: k[x_1, \dots, x_n][1/f] \rightarrow A$ satisfying $\psi \circ \varphi = 1_A$;
- (5) K is said to be *k-unirational* if $k \subset K \subset k(x_1, \dots, x_n)$ for some integer n .

In Saltman's original definition of retract *k*-rationality ([Sal82b, page 130], [Sal84b, Definition 3.1]), a base field k is required to be infinite in order to guarantee the existence of sufficiently many *k*-specializations. We now assume that k is an infinite field. Then if K and L are stably *k*-isomorphic and K is retract *k*-rational, then L is also retract *k*-rational (see [Sal84b, Proposition 3.6]), and it is not difficult to verify the following implications:

$$k\text{-rational} \Rightarrow \text{stably } k\text{-rational} \Rightarrow \text{retract } k\text{-rational} \Rightarrow k\text{-unirational}.$$

Note that $k(G)$ is retract *k*-rational if and only if there exists a generic G -Galois extension over k (see [Sal82a, Theorem 5.3], [Sal84b, Theorem 3.12]). In particular, if k is a Hilbertian field, e.g. number field, and $k(G)$ is retract *k*-rational, then inverse Galois problem for G over k has a positive answer, i.e. there exists a Galois extension K/k with $\text{Gal}(K/k) \simeq G$.

Swan [Swa69] gave the first negative solution to Noether's problem. He proved that if $p = 47, 113$ or 233 , then $\mathbb{Q}(C_p)$ is not \mathbb{Q} -rational, where C_p is the cyclic group of order prime p , by using Masuda's idea of Galois descent [Mas55, Mas68].

Noether's problem for abelian groups was studied extensively by Masuda, Kuniyoshi, Swan, Voskresenskii, Endo and Miyata, etc. Eventually, Lenstra [Len74] gave a necessary and sufficient condition to Noether's problem for finite abelian groups. For details, see Swan's survey paper [Swa83], Voskresenskii's book [Vos98, Section 7] or [Hos]. On the other hand, just a handful of results about Noether's problem are obtained when the groups are non-abelian.

Theorem 1.4 (Maeda [Mae89, Theorem, page 418]). *Let k be a field and A_5 be the alternating group of degree 5. Then $k(A_5)$ is *k*-rational.*

Theorem 1.5 (Serre [GMS03, Chapter IX], see also Kang [Kan05]). *Let G be a group with a 2-Sylow subgroup which is cyclic of order ≥ 8 or the generalized quaternion Q_{16} of order 16. Then $\mathbb{Q}(G)$ is not stably \mathbb{Q} -rational.*

Theorem 1.6 (Plans [Pla09, Theorem 2]). *Let A_n be the alternating group of degree n . If $n \geq 3$ is odd integer, then $\mathbb{Q}(A_n)$ is rational over $\mathbb{Q}(A_{n-1})$. In particular, if $\mathbb{Q}(A_{n-1})$ is \mathbb{Q} -rational, then so is $\mathbb{Q}(A_n)$.*

However, it is an open problem whether $k(A_n)$ is *k*-rational for $n \geq 6$.

From now on, we restrict ourselves to the case where G is a p -group. By Theorem 1.1 and Theorem 1.2, we may focus on the case where G is a non-abelian p -group and k is a field with $\text{char } k \neq p$. For p -groups of small order, the following results are known.

Theorem 1.7 (Chu and Kang [CK01]). *Let p be any prime and G be a p -group of order $\leq p^4$ and of exponent e . If k is a field containing a primitive e -th root of unity, then $k(G)$ is k -rational.*

Theorem 1.8 (Chu, Hu, Kang and Prokhorov [CHKP08]). *Let G be a group of order 32 and of exponent e . If k is a field containing a primitive e -th root of unity, then $k(G)$ is k -rational.*

For more recent results, see e.g. [HK10], [Kan11], [KMZ12].

Saltman introduced a notion of retract k -rationality (see Definition 1.3) and the unramified Brauer group. Recall that the implications for an infinite field k : k -rational \Rightarrow stably k -rational \Rightarrow retract k -rational. Hence if $k(G)$ is not retract k -rational, then it is not k -rational.

Definition 1.9 (Saltman [Sal84a, Definition 3.1], [Sal85, page 56]). Let K/k be an extension of fields. The *unramified Brauer group* $\text{Br}_{\text{nr}}(K/k)$ of K over k is defined to be

$$\text{Br}_{\text{nr}}(K/k) = \bigcap_R \text{Image}\{\text{Br}(R) \rightarrow \text{Br}(K)\}$$

where $\text{Br}(R) \rightarrow \text{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R . We omit k from the notation and write just $\text{Br}_{\text{nr}}(K)$ when the base field k is clear from the context.

Proposition 1.10 (Saltman [Sal84a], [Sal85, Proposition 1.8], [Sal87]). *If K is retract k -rational, then $\text{Br}(k) \xrightarrow{\sim} \text{Br}_{\text{nr}}(K)$. In particular, if k is an algebraically closed field and K is retract k -rational, then $\text{Br}_{\text{nr}}(K) = 0$.*

Theorem 1.11 (Bogomolov [Bog88, Theorem 3.1], Saltman [Sal90, Theorem 12]). *Let G be a finite group and k be an algebraically closed field with $\text{char } k = 0$ or $\text{char } k = p$ with $p \nmid |G|$. Then $\text{Br}_{\text{nr}}(k(G)/k)$ is isomorphic to the group $B_0(G)$ defined by*

$$B_0(G) = \bigcap_A \text{Ker}\{\text{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

Remark 1.12. For a smooth projective variety X over \mathbb{C} with function field K , $\text{Br}_{\text{nr}}(K/\mathbb{C})$ is isomorphic to the birational invariant $H^3(X, \mathbb{Z})_{\text{tors}}$ which was used by Artin and Mumford [AM72] to provide some elementary examples of k -unirational varieties which are not k -rational (see [Bog88, Theorem 1.1 and Corollary]).

Following Kunyavskii [Kun10], we call $B_0(G)$ the Bogomolov multiplier of G . Note that $B_0(G)$ is a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})$ which is isomorphic to the Schur multiplier $H_2(G, \mathbb{Z})$ of G (see Karpilovsky [Kar87]). Because of Theorem 1.11, we will not distinguish $B_0(G)$ and $\text{Br}_{\text{nr}}(k(G)/k)$ when k is an algebraically closed field, and $\text{char } k = 0$ or $\text{char } k = p$ with $p \nmid |G|$.

Using the Bogomolov multiplier $B_0(G)$, Saltman and Bogomolov gave counter-examples to Noether's problem for non-abelian p -groups over algebraically closed field.

Theorem 1.13 (Saltman [Sal84a], Bogomolov [Bog88]). *Let p be any prime and k be any algebraically closed field with $\text{char } k \neq p$.*

- (1) (Saltman [Sal84a, Theorem 3.6]) *There exists a meta-abelian group G of order p^9 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not (retract, stably) k -rational;*
- (2) (Bogomolov [Bog88, Lemma 5.6]) *There exists a group G of order p^6 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not (retract, stably) k -rational.*

Colliot-Thélène and Ojanguren [CTO89] generalized the notion of the unramified Brauer group $\text{Br}_{\text{nr}}(K/k)$ to the unramified cohomology $H_{\text{nr}}^i(K/k, \mu_n^{\otimes j})$ of degree $i \geq 1$, that is $F_n^{i,j}(K/k)$ in [CTO89, Definition 1.1].

Definition 1.14 (Colliot-Thélène and Ojanguren [CTO89], see also [CT95, Sections 2–4]). Let n be a positive integer and k be an algebraically closed field with $\text{char } k = 0$ or $\text{char } k = p$ with $p \nmid n$. Let K/k be a function field, that is finitely generated as a field over k . The *unramified cohomology group* $H_{\text{nr}}^i(K/k, \mu_n^{\otimes j})$ of K over k of degree $i \geq 1$ is defined to be

$$H_{\text{nr}}^i(K/k, \mu_n^{\otimes j}) = \bigcap_R \text{Image}\{H_{\text{ét}}^i(R, \mu_n^{\otimes j}) \rightarrow H_{\text{ét}}^i(K, \mu_n^{\otimes j})\}$$

where R runs over all the discrete valuation rings R of rank one such that $k \subset R \subset K$ and K is the quotient field of R . We write just $H_{\text{nr}}^i(K, \mu_n^{\otimes j})$ when the base field k is clear.

Note that the unramified cohomology groups of degree two are isomorphic to the n -torsion part of the unramified Brauer group: ${}_n\text{Br}_{\text{nr}}(K/k) \simeq H_{\text{nr}}^2(K/k, \mu_n)$.

Proposition 1.15. *Let k be an algebraically closed field with $\text{char } k = 0$ or $\text{char } k = p$ with $p \nmid n$.*
 (1) (Colliot-Thélène and Ojanguren [CTO89, Proposition 1.2]) *If K and L are stably k -isomorphic, then $H_{\text{nr}}^i(K/k, \mu_n^{\otimes j}) \xrightarrow{\sim} H_{\text{nr}}^i(L/k, \mu_n^{\otimes j})$. In particular, K is stably k -rational, then $H_{\text{nr}}^i(K/k, \mu_n^{\otimes j}) = 0$;*
 (2) ([Mer08, Proposition 2.15], see also [CTO89, Remarque 1.2.2], [CT95, Sections 2–4], [GS10, Example 5.9]) *If K is retract k -rational, then $H_{\text{nr}}^i(K/k, \mu_n^{\otimes j}) = 0$.*

Colliot-Thélène and Ojanguren [CTO89, Section 3] produced the first example of not stably \mathbb{C} -rational but \mathbb{C} -unirational field K with $H_{\text{nr}}^3(K, \mu_2^{\otimes 3}) \neq 0$, where K is the function field of a quadric of the type $\langle\langle f_1, f_2 \rangle\rangle = \langle g_1 g_2 \rangle$ over the rational function field $\mathbb{C}(x, y, z)$ with three variables x, y, z for a 2-fold Pfister form $\langle\langle f_1, f_2 \rangle\rangle$, as a generalization of Artin and Mumford [AM72]. Peyre [Pey93, Corollary 3] gave a sufficient condition for $H_{\text{nr}}^i(K/k, \mu_p^{\otimes i}) \neq 0$ and produced an example of the function field K with $H_{\text{nr}}^3(K/k, \mu_p^{\otimes 3}) \neq 0$ and $\text{Br}_{\text{nr}}(K/k) = 0$ using a result of Susulin [Sus91] where K is the function field of a product of some norm varieties associated to cyclic central simple algebras of degree p (see [Pey93, Proposition 7]). Using a result of Jacob and Rost [JR89], Peyre [Pey93, Proposition 9] also gave an example of $H_{\text{nr}}^4(K/k, \mu_2^{\otimes 4}) \neq 0$ and $\text{Br}_{\text{nr}}(K/k) = 0$ where K is the function field of a product of quadrics associated to a 4-fold Pfister form $\langle\langle a_1, a_2, a_3, a_4 \rangle\rangle$ (see also [CT95, Section 4.2]).

Take the direct limit with respect to n :

$$H^i(K/k, \mathbb{Q}/\mathbb{Z}(j)) = \varinjlim_n H^i(K/k, \mu_n^{\otimes j})$$

and we also define the unramified cohomology group

$$H_{\text{nr}}^i(K/k, \mathbb{Q}/\mathbb{Z}(j)) = \bigcap_R \text{Image}\{H_{\text{ét}}^i(R, \mathbb{Q}/\mathbb{Z}(j)) \rightarrow H_{\text{ét}}^i(K, \mathbb{Q}/\mathbb{Z}(j))\}.$$

Then we have $\text{Br}_{\text{nr}}(K/k) \simeq H_{\text{nr}}^2(K/k, \mathbb{Q}/\mathbb{Z}(1))$.

Peyre [Pey08] was able to construct an example of a field K , as $K = \mathbb{C}(G)$, whose unramified Brauer group vanishes, but unramified cohomology of degree three does not vanish:

Theorem 1.16 (Peyre [Pey08, Theorem 3]). *Let p be any odd prime. There exists a p -group G of order p^{12} such that $B_0(G) = 0$ and $H_{\text{nr}}^3(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) \mathbb{C} -rational.*

Asok [Aso13] generalized Peyre's argument [Pey93] and established the following theorem for a smooth proper model X (resp. a smooth projective model Y) of the function field of a product of quadrics of the type $\langle\langle s_1, \dots, s_n \rangle\rangle = \langle s_n \rangle$ (resp. Rost varieties) over some rational function field over \mathbb{C} with many variables.

Theorem 1.17 (Asok [Aso13], see also [AM11, Theorem 3] for retract \mathbb{C} -rationality).

- (1) ([Aso13, Theorem 1]) *For any $n > 0$, there exists a smooth projective complex variety X that is \mathbb{C} -unirational, for which $H_{\text{nr}}^i(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H_{\text{nr}}^n(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so X is not \mathbb{A}^1 -connected, nor (retract, stably) \mathbb{C} -rational;*
- (2) ([Aso13, Theorem 3]) *For any prime l and any $n \geq 2$, there exists a smooth projective rationally connected complex variety Y such that $H_{\text{nr}}^n(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$. In particular, Y is not \mathbb{A}^1 -connected, nor (retract, stably) \mathbb{C} -rational.*

Namely, the triviality of the unramified Brauer group or the unramified cohomology of higher degree is just a necessary condition of \mathbb{C} -rationality of fields. It is unknown whether the vanishing of all the unramified cohomologies is a sufficient condition for \mathbb{C} -rationality. It is interesting to consider an analog of Theorem 1.17 for quotient varieties V/G , e.g. $\mathbb{C}(V_{\text{reg}}/G) = \mathbb{C}(G)$.

The case where G is a group of order p^5 ($p \geq 3$).

From Theorem 1.13 (2), Bogomolov [Bog88, Remark 1] raised a question to classify the groups of order p^6 with $B_0(G) \neq 0$. He also claimed that if G is a p -group of order $\leq p^5$, then $B_0(G) = 0$ ([Bog88, Lemma 5.6]). However, this claim was disproved by Moravec:

Theorem 1.18 (Moravec [Mor12, Section 8]). *Let G be a group of order 243. Then $B_0(G) \neq 0$ if and only if $G = G(3^5, i)$ with $28 \leq i \leq 30$, where $G(3^5, i)$ is the i -th group of order 243 in the GAP database [GAP]. Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_3$.*

Moravec [Mor12] gave a formula for $B_0(G)$ by using a nonabelian exterior square $G \wedge G$ of G and an implemented algorithm **b0g.g** in computer algebra system GAP [GAP], which is available from his website www.fmf.uni-lj.si/~moravec/b0g.g. The number of all solvable groups G of order ≤ 729 apart from the orders 512, 576 and 640 with $B_0(G) \neq 0$ was given as in [Mor12, Table 1].

Hoshi, Kang and Kunyavskii [HKK13] determined p -groups G of order p^5 with $B_0(G) \neq 0$ for any $p \geq 3$. It turns out that they belong to the same isoclinism family.

Definition 1.19 (Hall [Hal40, page 133]). Let G be a finite group. Let $Z(G)$ be the center of G and $[G, G]$ be the commutator subgroup of G . Two p -groups G_1 and G_2 are called *isoclinic* if there exist group isomorphisms $\theta: G_1/Z(G_1) \rightarrow G_2/Z(G_2)$ and $\phi: [G_1, G_1] \rightarrow [G_2, G_2]$ such that $\phi([g, h]) = [g', h']$ for any $g, h \in G_1$ with $g' \in \theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$:

$$\begin{array}{ccc} G_1/Z_1 \times G_1/Z_1 & \xrightarrow{(\theta, \theta)} & G_2/Z_2 \times G_2/Z_2 \\ \downarrow [\cdot, \cdot] & \circlearrowleft & \downarrow [\cdot, \cdot] \\ [G_1, G_1] & \xrightarrow{\phi} & [G_2, G_2]. \end{array}$$

For a prime p and an integer n , we denote by $G_n(p)$ the set of all non-isomorphic groups of order p^n . In $G_n(p)$, consider an equivalence relation: two groups G_1 and G_2 are equivalent if and only if they are isoclinic. Each equivalence class of $G_n(p)$ is called an *isoclinism family*, and the j -th isoclinism family is denoted by Φ_j .

For $p \geq 5$ (resp. $p = 3$), there exist $2p + 61 + \gcd\{4, p - 1\} + 2\gcd\{3, p - 1\}$ (resp. 67) groups G of order p^5 which are classified into ten isoclinism families Φ_1, \dots, Φ_{10} (see [Jam80, Section 4]). The main theorem of [HKK13] can be stated as follows:

Theorem 1.20 (Hoshi, Kang and Kunyavskii [HKK13, Theorem 1.12], [Kan14, page 424]). *Let p be any odd prime and G be a group of order p^5 . Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10} . Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_p$.*

For the last statement, see [Kan14, Remark, page 424]. The proof of Theorem 1.20 was given by purely algebraic way. There exist exactly 3 groups which belong to Φ_{10} if $p = 3$, i.e. $G = G(243, i)$ with $28 \leq i \leq 30$. This agrees with Moravec's computational result (Theorem 1.18). For $p \geq 5$, there exist exactly $1 + \gcd\{4, p - 1\} + \gcd\{3, p - 1\}$ groups which belong to Φ_{10} ([Jam80, page 621]).

The following result for the k -rationality of $k(G)$ supplements Theorem 1.18 although it is unknown whether $k(G)$ is k -rational for groups G which belong to Φ_7 :

Theorem 1.21 (Chu, Hoshi, Hu and Kang [CHHK, Theorem 1.13]). *Let G be a group of order 243 with exponent e . If $B_0(G) = 0$ and k be a field containing a primitive e -th root of unity, then $k(G)$ is k -rational except possibly for the five groups G which belong to Φ_7 , i.e. $G = G(243, i)$ with $56 \leq i \leq 60$.*

In [HKK13] and [CHHK], not only the evaluation of the Bogomolov multiplier $B_0(G)$ and the k -rationality of $k(G)$ but also the k -isomorphisms between $k(G_1)$ and $k(G_2)$ for some groups G_1 and G_2 belonging to the same isoclinism family were given.

Bogomolov and Böhning [BB13] gave an answer to the question raised as [HKK13, Question 1.11] in the affirmative as follows.

Theorem 1.22 (Bogomolov and Böhning [BB13, Theorem 6]). *If G_1 and G_2 are isoclinic, then $\mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably \mathbb{C} -isomorphic. In particular, $H_{\text{nr}}^i(\mathbb{C}(G_1), \mu_n^{\otimes j}) \xrightarrow{\sim} H_{\text{nr}}^i(\mathbb{C}(G_2), \mu_n^{\otimes j})$.*

A partial result of Theorem 1.22 was already given by Moravec. Indeed, Moravec [Mor14, Theorem 1.2] proved that if G_1 and G_2 are isoclinic, then $B_0(G_1) \simeq B_0(G_2)$.

The case where G is a group of order 64.

The classification of the groups G of order p^6 with $B_0(G) \neq 0$ for $p = 2$ was obtained by Chu, Hu, Kang and Kunyavskii [CHKK10]. Moreover, they investigated Noether's problem for groups G with $B_0(G) = 0$. There exist 267 groups G of order 64 which are classified into 27 isoclinism families Φ_1, \dots, Φ_{27} by Hall and Senior [HS64] (see also [JNO90, Table I]). The main result of [CHKK10] can be stated in terms of the isoclinism families as follows.

Theorem 1.23 (Chu, Hu, Kang and Kunyavskii [CHKK10]). *Let $G = G(2^6, i)$, $1 \leq i \leq 267$, be the i -th group of order 64 in the GAP database [GAP].*

- (1) ([CHKK10, Theorem 1.8]) *$B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16} , i.e. $G = G(2^6, i)$ with $149 \leq i \leq 151$, $170 \leq i \leq 172$, $177 \leq i \leq 178$ or $i = 182$. Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq C_2$ (see [Kan14, Remark, page 424] for this statement);*
- (2) ([CHKK10, Theorem 1.10]) *If $B_0(G) = 0$ and k is an quadratically closed field, then $k(G)$ is k -rational except possibly for five groups which belong to Φ_{13} , i.e. $G = G(2^6, i)$ with $241 \leq i \leq 245$.*

For groups G which belong to Φ_{13} , k -rationality of $k(G)$ is unknown. The following two propositions supplement the cases Φ_{13} and Φ_{16} of Theorem 1.23.

Definition 1.24. Let k be a field with $\text{char } k \neq 2$ and $k(X_1, X_2, X_3, X_4, X_5, X_6)$ be the rational function field over k with variables $X_1, X_2, X_3, X_4, X_5, X_6$.

(i) The field $L_k^{(0)}$ is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6)^H$ where $H = \langle \sigma_1, \sigma_2 \rangle \simeq C_2 \times C_2$ act on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ by k -automorphisms

$$\begin{aligned}\sigma_1 : X_1 &\mapsto X_3, X_2 \mapsto \frac{1}{X_1 X_2 X_3}, X_3 \mapsto X_1, X_4 \mapsto X_6, X_5 \mapsto \frac{1}{X_4 X_5 X_6}, X_6 \mapsto X_4, \\ \sigma_2 : X_1 &\mapsto X_2, X_2 \mapsto X_1, X_3 \mapsto \frac{1}{X_1 X_2 X_3}, X_4 \mapsto X_5, X_5 \mapsto X_4, X_6 \mapsto \frac{1}{X_4 X_5 X_6}.\end{aligned}$$

(ii) The field $L_k^{(1)}$ is defined to be $k(X_1, X_2, X_3, X_4)^{\langle \tau \rangle}$ where $\langle \tau \rangle \simeq C_2$ acts on $k(X_1, X_2, X_3, X_4)$ by k -automorphisms

$$\tau : X_1 \mapsto -X_1, X_2 \mapsto \frac{X_4}{X_2}, X_3 \mapsto \frac{(X_4 - 1)(X_4 - X_1^2)}{X_3}, X_4 \mapsto X_4.$$

Proposition 1.25 ([CHKK10, Proposition 6.3], see also [HY, Proposition 12.5]). *Let G be a group of order 64 which belongs to Φ_{13} , i.e. $G = G(2^6, i)$ with $241 \leq i \leq 245$. There exists a \mathbb{C} -injective homomorphism $\varphi : L_{\mathbb{C}}^{(0)} \rightarrow \mathbb{C}(G)$ such that $\mathbb{C}(G)$ is rational over $\rho(L_{\mathbb{C}}^{(0)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(0)}$ are stably \mathbb{C} -isomorphic and $B_0(G) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(0)}) = 0$.*

Proposition 1.26 ([CHKK10, Example 5.11, page 2355], [HKK14, Proof of Theorem 6.3]). *Let G be a group of order 64 which belongs to Φ_{16} , i.e. $G = G(2^6, i)$ with $149 \leq i \leq 151$, $170 \leq i \leq 172$, $177 \leq i \leq 178$ or $i = 182$. There exists a \mathbb{C} -injective homomorphism $\varphi : L_{\mathbb{C}}^{(1)} \rightarrow \mathbb{C}(G)$ such that $\mathbb{C}(G)$ is rational over $\varphi(L_{\mathbb{C}}^{(1)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are stably \mathbb{C} -isomorphic, $B_0(G) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(1)}) \simeq C_2$ and hence $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are not (retract, stably) \mathbb{C} -rational.*

Proof. The case of $G = G(2^6, 149)$ is given in [HKK14, Proof of Theorem 6.3], see also [CHKK10, Example 5.11, page 2355]. The proof for other cases is similar. \square

Question 1.27 ([CHKK10, Section 6], see also [HY, Section 12]). *Is $L_k^{(0)}$ k -rational?*

The case where G is a group of order 128.

There exist 2328 groups of order 128 which are classified into 115 isoclinism families $\Phi_1, \dots, \Phi_{115}$ ([JNO90, Tables I, II, III]). By using Moravec's algorithm **b0g.g** [Mor12] of GAP [GAP], e.g. "for i in [1..2328] do Print([i, B0G(SmallGroup(128, i))], "\n"); od;", we obtain the following theorem.

Theorem 1.28 (Moravec [Mor12, Section 8, Table 1]). *Let G be a group of order 128. Then $B_0(G) \neq 0$ if and only if G is one of the following 220 groups:*

- (1) $G(2^7, i)$ with $i = 227, 228, 229, 301, 324, 325, 326, 541, 543, 568, 570, 579, 581, 626, 627, 629, 667, 668, 670, 675, 676, 678, 691, 692, 693, 695, 703, 704, 705, 707, 724, 725, 727, 1783, 1784, 1785, 1786, 1864, 1865, 1866, 1867, 1880, 1881, 1882, 1893, 1894, 1903, 1904$;
- (2) $G(2^7, i)$ with $1345 \leq i \leq 1399$;
- (3) $G(2^7, i)$ with $242 \leq i \leq 247$, $265 \leq i \leq 269$, $287 \leq i \leq 293$;
- (4) $G(2^7, i)$ with $36 \leq i \leq 41$;
- (5) $G(2^7, i)$ with $1924 \leq i \leq 1929$, $1945 \leq i \leq 1951$, $1966 \leq i \leq 1972$, $1983 \leq i \leq 1988$;
- (6) $G(2^7, i)$ with $417 \leq i \leq 436$;
- (7) $G(2^7, i)$ with $446 \leq i \leq 455$;
- (8) $G(2^7, i)$ with $i = 950, 951, 952, 975, 976, 977, 982, 983, 987$;
- (9) $G(2^7, i)$ with $i = 144, 145$;
- (10) $G(2^7, i)$ with $i = 138, 139$;
- (11) $G(2^7, i)$ with $1544 \leq i \leq 1577$.

Moreover, if G is a group in (1)–(10) (resp. (11)), then $B_0(G) \simeq C_2$ (resp. $C_2 \times C_2$).

By [JNO90, Tables I, II, III], we can get the classification of 115 isoclinism families for groups G of order 128 in terms of the GAP database [GAP]. We will present the complete classification as Table 2 in Section 3. Using this, we see that the groups as in (1)–(11) of Theorem 1.28 correspond to the isoclinism families $\Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106}, \Phi_{114}, \Phi_{30}$ respectively:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	Total
Family	Φ_{16}	Φ_{31}	Φ_{37}	Φ_{39}	Φ_{43}	Φ_{58}	Φ_{60}	Φ_{80}	Φ_{106}	Φ_{114}	Φ_{30}	
$\exp(G)$	8	4	8	4 or 8	8	8	8	16	8	8	4	
$B_0(G)$	C_2										$C_2 \times C_2$	
# G 's	48	55	18	6	26	20	10	9	2	2	34	220

Table 1: Isoclinism families Φ_j for groups G of order 128 with $B_0(G) \neq 0$

Corollary 1.29 (Moravec [Mor12, Section 8, Table 1]). *Let G be a group of order 128. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family $\Phi_{16}, \Phi_{30}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106}$ or Φ_{114} . Moreover, if $B_0(G) \neq 0$, then*

$$B_0(G) \simeq \begin{cases} C_2 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106} \text{ or } \Phi_{114}, \\ C_2 \times C_2 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\mathbb{C}(G)$ is not (retract, stably) \mathbb{C} -rational.

The aim of this paper is to investigate the birational classification of $\mathbb{C}(G)$ for groups G of order 128. In particular, we consider what happens to $\mathbb{C}(G)$ with $B_0(G) \neq 0$? The main theorem (Theorem 1.31) of this paper gives a partial answer to this question.

Definition 1.30. Let k be a field with $\text{char } k \neq 2$ and $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ be the rational function field over k with variables $X_1, X_2, X_3, X_4, X_5, X_6, X_7$.

(i) *The field $L_k^{(2)}$ is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle \rho \rangle}$ where $\langle \rho \rangle \simeq C_4$ acts on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ by k -automorphisms*

$$\begin{aligned} \rho : X_1 &\mapsto X_2, X_2 \mapsto -X_1, X_3 \mapsto X_4, X_4 \mapsto X_3, \\ X_5 &\mapsto X_6, X_6 \mapsto \frac{(X_1^2 X_2^2 - 1)(X_1^2 X_3^2 + X_2^2 - X_3^2 - 1)}{X_5}. \end{aligned}$$

(ii) *The field $L_k^{(3)}$ is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)^{\langle \lambda_1, \lambda_2 \rangle}$ where $\langle \lambda_1, \lambda_2 \rangle \simeq C_2 \times C_2$ acts on $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ by k -automorphisms*

$$\begin{aligned} \lambda_1 : X_1 &\mapsto X_1, X_2 \mapsto \frac{X_1}{X_2}, X_3 \mapsto \frac{1}{X_1 X_3}, X_4 \mapsto \frac{X_2 X_4}{X_1 X_3}, \\ X_5 &\mapsto -\frac{X_1 X_6^2 - 1}{X_5}, X_6 \mapsto -X_6, X_7 \mapsto X_7, \\ \lambda_2 : X_1 &\mapsto \frac{1}{X_1}, X_2 \mapsto X_3, X_3 \mapsto X_2, X_4 \mapsto \frac{(X_1 X_6^2 - 1)(X_1 X_7^2 - 1)}{X_4}, \\ X_5 &\mapsto -X_5, X_6 \mapsto -X_1 X_6, X_7 \mapsto -X_1 X_7. \end{aligned}$$

Theorem 1.31 (see Theorem 2.1). *Let G be a group of order 128. Assume that $B_0(G) \neq 0$. Then $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(m)}$ are stably \mathbb{C} -isomorphic where*

$$m = \begin{cases} 1 & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60} \text{ or } \Phi_{80}, \\ 2 & \text{if } G \text{ belongs to } \Phi_{106} \text{ or } \Phi_{114}, \\ 3 & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(1)}) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(2)}) \simeq C_2$ and $\text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(3)}) \simeq C_2 \times C_2$ and hence the fields $L_{\mathbb{C}}^{(1)}$, $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$ are not (retract, stably) \mathbb{C} -rational.

For $m = 1, 2$, the fields $L_{\mathbb{C}}^{(m)}$ and $L_{\mathbb{C}}^{(3)}$ are not stably \mathbb{C} -isomorphic because their unramified Brauer groups are not isomorphic. However, we do not know whether the fields $L_{\mathbb{C}}^{(1)}$ and $L_{\mathbb{C}}^{(2)}$ are stably \mathbb{C} -isomorphic. If not, it is interesting to evaluate the higher unramified cohomologies. Unfortunately, a useful formula like Bogomolov's formula (Theorem 1.11) or Moravec's formula [Mor12, Section 3] for $B_0(G)$ is unknown for higher unramified cohomologies.

Theorem 1.31 gives another proof of $B_0(G) \simeq C_2$ to Theorem 1.28 when G belongs to $\Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}$ or Φ_{80} . Especially, this proof is based on the result of order 64 for Φ_{16} (Theorem 1.23) and it does not depend on the computer calculations of GAP.

Although Theorem 1.31 gives only the first step, the author hopes that it will stimulate further work towards a more complete understanding of the (stably) birational classification of $\mathbb{C}(G)$ for non-abelian groups G .

Standing Notations. Throughout this paper, G is a finite group and k is a base field. $Z(G)$ denotes the center of G . For $g, h \in G$, define the commutator $[g, h] = g^{-1}h^{-1}gh$. The exponent of a group G is defined to be $\text{lcm}\{\text{ord}(g) : g \in G\}$ where $\text{ord}(g)$ is the order of the element g .

We denote by ζ_n a primitive n -th root of unity in a fixed algebraic closure of k . Whenever we write $\zeta_n \in k$, it is understood that either $\text{char } k = 0$ or $\text{char } k = p$ with $p \nmid n$. We will write ζ for ζ_4 for simplicity, η for a primitive 8th root of unity ζ_8 satisfying $\eta^2 = \zeta$ and ω for a primitive 16th root of unity ζ_{16} satisfying $\omega^2 = \eta$.

The group $G(2^7, i)$ of order 128, or $G(i)$ for short, is the i -th group of order 128 in the GAP database [GAP]. The version of GAP used in this paper is GAP4, Version: 4.4.12 [GAP].

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2. PROOF OF THEOREM 1.31

We will prove Theorem 1.31. By Bogomolov and Böhning's theorem (Theorem 1.22), we may choose one group G within each isoclinism family Φ_j . We will choose the first one, i.e. $G = G(2^7, i) = G(i)$ with the minimal i of the GAP database within each isoclinism family (see also Table 2 in Section 3). More precisely, we will show the following theorem.

Theorem 2.1. *Let $G = G(i)$ be the i -th group of order 128 with exponent e in the GAP database [GAP]. Let k be a field with $\text{char } k \neq 2$ and $\zeta_e \in k$.*

- (i) *If G is one of the groups $G(227)$, $G(1345)$, $G(242)$, $G(36)$, $G(1924)$, $G(417)$, $G(446)$ and $G(950)$ which belong to the isoclinism families Φ_{16} , Φ_{31} , Φ_{37} , Φ_{39} , Φ_{43} , Φ_{58} , Φ_{60} and Φ_{80} respectively, then there exists a k -injective homomorphism $\varphi : L_k^{(1)} \rightarrow k(G)$ such that $k(G)$ is rational over $\varphi(L_k^{(1)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are stably \mathbb{C} -isomorphic and $B_0(G) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(1)}) \simeq C_2$;*
- (ii) *If G is one of the groups $G(144)$ and $G(138)$ which belong to Φ_{106} and Φ_{114} respectively, then*

there exists a k -injective homomorphism $\varphi : L_k^{(2)} \rightarrow k(G)$ such that $k(G)$ is rational over $\varphi(L_k^{(2)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(2)}$ are stably \mathbb{C} -isomorphic and $B_0(G) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(2)}) \simeq C_2$;
 (iii) If $G = G(1544)$ which belongs to Φ_{30} , then there exists a k -injective homomorphism $\varphi : L_k^{(3)} \rightarrow k(G)$ such that $k(G)$ is rational over $\varphi(L_k^{(3)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(3)}$ are stably \mathbb{C} -isomorphic and $B_0(G) \simeq \text{Br}_{\text{nr}}(L_{\mathbb{C}}^{(3)}) \simeq C_2 \times C_2$.

Proof of Theorem 2.1.

We first prepare the following lemmas which will be used.

Theorem 2.2 (Hajja and Kang [HK95, Theorem 1]). *Let L be any field and G be a finite group acting on $L(x_1, \dots, x_n)$, the rational function field of n variables over L . Suppose that*

- (i) *for any $\sigma \in G$, $\sigma(L) \subset L$;*
- (ii) *the restriction of the action of G to L is faithful; and*
- (iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over L .

Then there exist $z_1, \dots, z_n \in L(x_1, \dots, x_n)$ such that $L(x_1, \dots, x_n) = L(z_1, \dots, z_n)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

Theorem 2.3 (Ahmad, Hajja and Kang [AHK00, Theorem 3.1]). *Let L be any field, $L(x)$ be the rational function field over L with variable x and G be a finite group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_{\sigma}x + b_{\sigma}$ where $a_{\sigma}, b_{\sigma} \in L$ and $a_{\sigma} \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$. In fact, if $m = \min\{\deg g(x) : g(x) \in L[x]^G \setminus L\}$, any polynomial $f \in L[x]^G$ with $\deg f = m$ satisfies the property $L(x)^G = L^G(f)$.*

Lemma 2.4 (Hoshi, Kitayama and Yamasaki [HKY11, Lemma 3.9]). *Let k be a field with $\text{char } k \neq 2$ and $\langle \tau_3 \rangle \simeq C_2$ act on the rational function field $k(x, y, z)$ over k with variables x, y, z by k -automorphisms*

$$\tau_3 : x \mapsto y \mapsto x, \quad z \mapsto \frac{c}{xyz} \mapsto z, \quad c \in k^{\times}.$$

Then $k(x, y, z)^{\langle \tau_3 \rangle} = k(t_1, t_2, t_3)$ where

$$t_1 = \frac{xy}{x+y}, \quad t_2 = \frac{xyz + \frac{c}{z}}{x+y}, \quad t_3 = \frac{xyz - \frac{c}{z}}{x-y}.$$

We will separate the proof of Theorem 2.1 (i), (ii), (iii) into Case 1 to Case 11:

- (i) Case 1: $G = G(2^7, 227)$ which belongs to Φ_{16} ;
 Case 2: $G = G(2^7, 1345)$ which belongs to Φ_{31} ;
 Case 3: $G = G(2^7, 242)$ which belongs to Φ_{37} ;
 Case 4: $G = G(2^7, 36)$ which belongs to Φ_{39} ;
 Case 5: $G = G(2^7, 1924)$ which belongs to Φ_{43} ;
 Case 6: $G = G(2^7, 417)$ which belongs to Φ_{58} ;
 Case 7: $G = G(2^7, 446)$ which belongs to Φ_{60} ;
 Case 8: $G = G(2^7, 950)$ which belongs to Φ_{80} ;
- (ii) Case 9: $G = G(2^7, 144)$ which belongs to Φ_{106} ;
 Case 10: $G = G(2^7, 138)$ which belongs to Φ_{114} ;
- (iii) Case 11: $G = G(2^7, 1544)$ which belongs to Φ_{30} .

The generators and the relations of the groups $G = G(2^7, i)$ can be found in the GAP database, e.g. `PrintPcpPresentation(PcGroupToPcpGroup(SmallGroup(2^7,i)))`.

Recall that $\zeta = \zeta_4$ is a primitive 4th root of unity, η is a primitive 8th root of unity satisfying $\eta^2 = \zeta$ and ω is a primitive 16th root of unity satisfying $\omega^2 = \eta$.

Case 1: $G = G(2^7, 227)$ which belongs to Φ_{16} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = g_5, g_2^2 = 1, g_3^2 = 1, g_4^2 = g_6, g_5^2 = g_7, g_6^2 = 1, g_7^2 = 1, Z(G) = \langle g_5, g_6, g_7 \rangle, [g_2, g_1] = g_4, [g_3, g_1] = g_7, [g_3, g_2] = g_6g_7, [g_4, g_1] = g_6, [g_4, g_2] = g_6$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{227}) \simeq GL_6(k)$ of dimension 6 which is decomposable into two irreducible components $V_{227} \simeq U_4 \oplus U_2$ of dimension 4 and 2 respectively. By Theorem 2.2, $k(G)$ is rational over $k(V_{227})^G$. Hence it is enough to show that $k(V_{227})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{227}) = k(y_1, y_2, y_3, y_4, y_5, y_6)$ is given by

$$\begin{aligned}
 g_1 : y_1 &\mapsto \zeta y_4, y_2 \mapsto -\zeta y_3, y_3 \mapsto -y_2, y_4 \mapsto y_1, y_5 \mapsto \eta y_6, y_6 \mapsto \eta y_5, \\
 g_2 : y_1 &\mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_6, y_6 \mapsto y_5, \\
 g_3 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto y_6, \\
 g_4 : y_1 &\mapsto -\zeta y_1, y_2 \mapsto -\zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, \\
 g_5 : y_1 &\mapsto \zeta y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto \zeta y_5, y_6 \mapsto \zeta y_6, \\
 g_6 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, \\
 g_7 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6.
 \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}, z_2 = \frac{y_2}{y_4}, z_3 = \frac{y_3}{y_4}, z_4 = \frac{y_5}{y_6}, z_5 = y_4, z_6 = y_6$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6) = k(z_1, z_2, z_3, z_4, z_5, z_6)$ and

$$\begin{aligned}
 g_1 : z_1 &\mapsto \frac{\zeta}{z_1}, z_2 \mapsto -\frac{\zeta z_3}{z_1}, z_3 \mapsto -\frac{z_2}{z_1}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto z_1 z_5, z_6 \mapsto \eta z_4 z_6, \\
 g_2 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto z_2 z_5, z_6 \mapsto z_4 z_6, \\
 g_3 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, \\
 g_4 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto z_6, \\
 g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto \zeta z_6, \\
 g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, \\
 g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto -z_6.
 \end{aligned}$$

Apply Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4)(z_5, z_6)$, $k(V_{227})^G = k(z_1, z_2, z_3, z_4, z_5, z_6)^G$ is rational over $k(z_1, z_2, z_3, z_4)^G$. We find that $k(z_1, z_2, z_3, z_4)^G = k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4 \rangle}$ because $Z(G) = \langle g_5, g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4)$ trivially. Thus it suffices to show that $k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = \frac{z_2 z_3}{z_1}$, $u_2 = \frac{z_1}{z_2 z_4}$, $u_3 = z_3 z_4$, $u_4 = \frac{z_3}{z_1 z_2}$. Then $k(u_1, u_2, u_3, u_4) \subset k(z_1, z_2, z_3, z_4)^{\langle g_3, g_4 \rangle}$ and the field extension degree $[k(z_1, z_2, z_3, z_4) : k(u_1, u_2, u_3, u_4)] = 4$ because the determinant of the matrix of exponents of u_1, u_2, u_3, u_4 with respect to z_1, z_2, z_3, z_4 is 4:

$$(1) \quad \det \begin{pmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} = 4.$$

Hence we have $k(z_1, z_2, z_3, z_4)^{\langle g_3, g_4 \rangle} = k(u_1, u_2, u_3, u_4)$ and

$$(2) \quad \begin{aligned} g_1 : u_1 &\mapsto u_1, u_2 \mapsto -\frac{1}{u_1 u_2}, u_3 \mapsto -\frac{u_1}{u_3}, u_4 \mapsto -\frac{1}{u_4}, \\ g_2 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto u_3, u_3 \mapsto u_2, u_4 \mapsto \frac{1}{u_4}. \end{aligned}$$

Define $v_1 = \left(\frac{u_1+1}{u_1-1}\right)\left(\frac{u_4+1}{u_4-1}\right)$, $v_2 = u_2 + u_3$, $v_3 = (u_2 - u_3)\left(\frac{u_1+1}{u_1-1}\right)$, $v_4 = \left(\frac{u_4+1}{u_4-1}\right)/\left(\frac{u_1+1}{u_1-1}\right)$. Then $k(u_1, u_2, u_3, u_4)^{\langle g_2 \rangle} = k(v_1, v_2, v_3, v_4)$ and

$$g_1 : v_1 \mapsto -\frac{1}{v_4}, v_2 \mapsto -\frac{4v_1(v_1 v_2 + v_2 v_4 + 2v_3 v_4)}{(v_1 - v_4)(v_1 v_2^2 - v_3^2 v_4)}, v_3 \mapsto \frac{4v_1(2v_1 v_2 + v_1 v_3 + v_3 v_4)}{(v_1 - v_4)(v_1 v_2^2 - v_3^2 v_4)}, v_4 \mapsto -\frac{1}{v_1}.$$

Define $w_1 = v_1$, $w_2 = v_1 v_2 + v_2 v_4 + 2v_3 v_4$, $w_3 = 2v_1 v_2 + v_1 v_3 + v_3 v_4$, $w_4 = \frac{v_1}{v_4}$. Then $k(v_1, v_2, v_3, v_4) = k(w_1, w_2, w_3, w_4)$ and

$$g_1 : w_1 \mapsto -\frac{w_4}{w_1}, w_2 \mapsto -\frac{4(w_4-1)(w_2-2w_3+w_2 w_4)}{w_2^2 w_4 - w_3^2}, w_3 \mapsto \frac{4(w_4-1)(w_3-2w_2 w_4 + w_3 w_4)}{w_2^2 w_4 - w_3^2}, w_4 \mapsto w_4.$$

We also define $X_1 = \frac{w_2 w_4 - w_3}{w_2 - w_3}$, $X_2 = \zeta w_1$, $X_3 = \frac{(w_2 - 2w_3 + w_2 w_4)(w_2^2 w_4 - w_3^2)}{2w_2(w_2 - w_3)(w_4 - 1)}$, $X_4 = w_4$. Then $k(w_1, w_2, w_3, w_4) = k(X_1, X_2, X_3, X_4)$ and

$$g_1 : X_1 \mapsto -X_1, X_2 \mapsto \frac{X_4}{X_2}, X_3 \mapsto \frac{(X_4-1)(X_4-X_1^2)}{X_3}, X_4 \mapsto X_4.$$

This action of g_1 on $k(X_1, X_2, X_3, X_4)$ and that of τ on $k(X_1, X_2, X_3, X_4)$ in Definition 1.24 (ii) are exactly the same. Hence $k(V_{227})^G$ is rational over $\varphi(L_k^{(1)})$.

Case 2: $G = G(2^7, 1345)$ which belongs to Φ_{31} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = g_2^2 = g_3^2 = g_4^2 = g_5^2 = g_6^2 = g_7^2 = 1$, $Z(G) = \langle g_5, g_6, g_7 \rangle$, $[g_2, g_1] = g_5$, $[g_3, g_1] = g_6$, $[g_3, g_2] = g_7$, $[g_4, g_3] = g_5$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{1345}) \simeq GL_8(k)$ of dimension 8 which is decomposable into three irreducible components $V_{1345} \simeq U_4 \oplus U_2 \oplus U_2'$ of dimension 4, 2 and 2 respectively. By Theorem 2.2, $k(G)$ is rational over $k(V_{1345})^G$. We will show that $k(V_{1345})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{1345}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_6, y_6 \mapsto y_5, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_2 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_8, y_8 \mapsto y_7, \\ g_3 : y_1 &\mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto y_4, y_4 \mapsto y_3, y_5 \mapsto -y_5, y_6 \mapsto y_6, y_7 \mapsto -y_7, y_8 \mapsto y_8, \\ g_4 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_5 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_6 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_7 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_6}$, $z_5 = \frac{y_7}{y_8}$, $z_6 = y_4$, $z_7 = y_6$, $z_8 = y_8$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto z_5, z_6 \mapsto z_2 z_6, z_7 \mapsto z_4 z_7, z_8 \mapsto z_8, \\ g_2 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto \frac{1}{z_5}, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_5 z_8, \\ g_3 : z_1 &\mapsto \frac{z_2}{z_3}, z_2 \mapsto \frac{z_1}{z_3}, z_3 \mapsto \frac{1}{z_3}, z_4 \mapsto -z_4, z_5 \mapsto -z_5, z_6 \mapsto z_3 z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto -z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto -z_7, z_8 \mapsto z_8, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto -z_8. \end{aligned}$$

Apply Theorem 2.3 three times to $k(z_1, z_2, z_3, z_4, z_5)(z_6, z_7, z_8)$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5)^G$. We find that $k(z_1, z_2, z_3, z_4, z_5)^G = k(z_1, z_2, z_3, z_4, z_5)^{\langle g_1, g_2, g_3, g_4 \rangle}$ because $Z(G) = \langle g_5, g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5)$ trivially. It suffices to show that the invariant field $k(z_1, z_2, z_3, z_4, z_5)^{\langle g_1, g_2, g_3, g_4 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = \frac{z_1 z_2}{z_3}$, $u_2 = \frac{z_2 z_3}{z_1}$, $u_3 = \frac{z_1 z_3}{z_2}$, $u_4 = z_4$, $u_5 = \frac{z_5 + 1}{z_2(z_5 - 1)}$. Note that $\frac{u'_5 + 1}{u'_5 - 1} = z_5$ where $u'_5 = u_5 z_2$. Hence $k(u_1, u_2, u_3, u_4, u_5) = k(u_1, u_2, u_3, u_4, \frac{z_5}{z_2})$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = 4$, $k(z_1, z_2, z_3, z_4, z_5)^{\langle g_2, g_4 \rangle} = k(u_1, u_2, u_3, u_4, u_5)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto \frac{1}{u_2}, u_3 \mapsto u_3, u_4 \mapsto \frac{1}{u_4}, u_5 \mapsto u_1 u_2 u_5, \\ g_3 : u_1 &\mapsto u_1, u_2 \mapsto \frac{1}{u_2}, u_3 \mapsto \frac{1}{u_3}, u_4 \mapsto -u_4, u_5 \mapsto \frac{1}{u_1 u_5}. \end{aligned}$$

Define $v_1 = \left(\frac{u_1 + 1}{u_1 - 1}\right)^2$, $v_2 = \left(\frac{u_1 + 1}{u_1 - 1}\right)\left(\frac{u_2 + 1}{u_2 - 1}\right)$, $v_3 = \left(\frac{u_1 + 1}{u_1 - 1}\right)\left(\frac{u_2 + 1}{u_2 - 1}\right)\left(\frac{u_3 + 1}{u_3 - 1}\right)$, $v_4 = \left(\frac{u_1 + 1}{u_1 - 1}\right)\left(\frac{u_4 + 1}{u_4 - 1}\right)$, $v_5 = (u_1 u_2 + 1)u_5$. Then $k(u_1, u_2, u_3, u_4, u_5)^{\langle g_1 \rangle} = k(v_1, v_2, v_3, v_4, v_5)$ and

$$g_3 : v_1 \mapsto v_1, v_2 \mapsto -v_2, v_3 \mapsto v_3, v_4 \mapsto \frac{v_1}{v_4}, v_5 \mapsto \frac{4v_1(v_2 + 1)(v_2 - 1)}{(v_1 - 1)(v_2^2 - v_1)v_5}.$$

Define $X_1 = \frac{v_1}{v_2}$, $X_2 = v_4$, $X_3 = \frac{(v_1 - 1)(v_1 - v_2^2)v_5}{2v_2(v_2 + 1)}$, $X_4 = v_1$, $X_5 = v_3$. Then $k(v_1, v_2, v_3, v_4, v_5) = k(X_1, X_2, X_3, X_4, X_5)$ and

$$g_1 : X_1 \mapsto -X_1, X_2 \mapsto \frac{X_4}{X_2}, X_3 \mapsto \frac{(X_4 - 1)(X_4 - X_1^2)}{X_3}, X_4 \mapsto X_4, X_5 \mapsto X_5.$$

The action of g_1 on $k(X_1, X_2, X_3, X_4)$ and that of τ on $k(X_1, X_2, X_3, X_4)$ in Definition 1.24 (ii) are exactly the same. Because g_1 acts on $k(X_5)$ trivially, we obtain that $k(X_1, X_2, X_3, X_4, X_5)^{\langle g_1 \rangle} = k(X_1, X_2, X_3, X_4)^{\langle g_1 \rangle}(X_5)$. Hence $k(V_{1345})^G$ is rational over $\varphi(L_k^{(1)})$.

Case 3: $G = G(2^7, 242)$ which belongs to Φ_{37} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = g_5$, $g_2^2 = 1$, $g_3^2 = 1$, $g_4^2 = g_7$, $g_5^2 = 1$, $g_6^2 = 1$, $g_7^2 = 1$, $Z(G) = \langle g_6, g_7 \rangle$, $[g_2, g_1] = g_4$, $[g_3, g_1] = g_7$, $[g_4, g_1] = g_6$, $[g_4, g_2] = g_7$, $[g_5, g_2] = g_6g_7$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{242}) \simeq GL_8(k)$ of dimension 8 which is decomposable into two irreducible components $V_{242} \simeq U_4 \oplus U'_4$ of dimension 4. By Theorem 2.2, $k(G)$ is rational over $k(V_{242})^G$. We will show that $k(V_{242})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{242}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto -y_2, y_2 \mapsto y_1, y_3 \mapsto -\zeta y_4, y_4 \mapsto \zeta y_3, y_5 \mapsto -y_7, y_6 \mapsto y_8, y_7 \mapsto y_5, y_8 \mapsto y_6, \\ g_2 : y_1 &\mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_6, y_6 \mapsto y_5, y_7 \mapsto y_8, y_8 \mapsto y_7, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_4 : y_1 &\mapsto -\zeta y_1, y_2 \mapsto -\zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_5 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto y_6, y_7 \mapsto -y_7, y_8 \mapsto y_8, \\ g_6 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8, \\ g_7 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_8}$, $z_5 = \frac{y_6}{y_8}$, $z_6 = \frac{y_7}{y_8}$, $z_7 = y_4$, $z_8 = y_8$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto -\frac{z_2}{\zeta z_3}, z_2 \mapsto \frac{z_1}{\zeta z_3}, z_3 \mapsto -\frac{1}{z_3}, z_4 \mapsto -\frac{z_6}{z_5}, z_5 \mapsto \frac{1}{z_5}, z_6 \mapsto \frac{z_4}{z_5}, z_7 \mapsto \zeta z_3 z_7, z_8 \mapsto z_5 z_8, \\ g_2 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{z_5}{z_6}, z_5 \mapsto \frac{z_4}{z_6}, z_6 \mapsto \frac{1}{z_6}, z_7 \mapsto z_2 z_7, z_8 \mapsto z_6 z_8, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, z_7 \mapsto \zeta z_7, z_8 \mapsto z_8, \\ g_5 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto -z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto -z_8, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto -z_7, z_8 \mapsto z_8. \end{aligned}$$

Apply Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4, z_5, z_6)(z_7, z_8)$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5, z_6)^G$. We find that $k(z_1, z_2, z_3, z_4, z_5, z_6)^G = k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ because $Z(G) = \langle g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5, z_6)$ trivially. Hence it suffices to show that $k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = \frac{z_1 z_2^2 z_6}{z_3^2 z_5}$, $u_2 = \frac{z_3}{z_1^3 z_4}$, $u_3 = -\frac{z_3^2 z_5}{\zeta z_2^2 z_6}$, $u_4 = \frac{z_1 z_3}{z_2}$, $u_5 = \frac{z_1 z_2 + z_3}{z_1 z_2 - z_3}$, $u_6 = \frac{z_4 + z_5 z_6}{z_4 - z_5 z_6}$. Note that $\frac{u_5 + 1}{u_5 - 1} = \frac{z_1 z_2}{z_3}$ and $\frac{u_6 + 1}{u_6 - 1} = \frac{z_4}{z_5 z_6}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = -8$, $k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_3, g_4, g_5 \rangle} = k(u_1, u_2, u_3, u_4, u_5, u_6)$ and

$$\begin{aligned} (3) \quad g_1 : u_1 &\mapsto \frac{-1}{u_1 u_2 u_3}, u_2 \mapsto u_3, u_3 \mapsto u_2, u_4 \mapsto \frac{1}{u_4}, u_5 \mapsto \frac{1}{u_5}, u_6 \mapsto \frac{-1}{u_6}, \\ g_2 : u_1 &\mapsto u_2, u_2 \mapsto u_1, u_3 \mapsto \frac{-1}{u_1 u_2 u_3}, u_4 \mapsto u_4, u_5 \mapsto -u_5, u_6 \mapsto -u_6. \end{aligned}$$

By applying Lemma 2.4, we obtain $k(u_1, u_2, u_3, u_4, u_5, u_6)^{\langle g_2 \rangle} = k(v_1, v_2, v_3, v_4, v_5, v_6)$ where

$$v_1 = \frac{u_1 u_2}{u_1 + u_2}, v_2 = \frac{u_1 u_2 u_3 + \frac{-1}{u_3}}{u_1 + u_2}, v_3 = \frac{u_1 u_2 u_3 - \frac{1}{u_3}}{u_1 - u_2}, v_4 = u_4, v_5 = \frac{u_5}{u_1 - u_2}, v_6 = \frac{u_6}{u_1 - u_2}.$$

The action of g_1 on $k(v_1, v_2, v_3, v_4, v_5, v_6)$ are given by

$$g_1 : v_1 \mapsto \frac{v_2^2 - v_3^2}{4v_1 v_2 (v_3^2 + 1)}, v_2 \mapsto -\frac{1}{v_2}, v_3 \mapsto -\frac{1}{v_3}, v_4 \mapsto \frac{1}{v_4}, v_5 \mapsto \frac{v_2^2 - v_3^2}{4(v_2^2 + 1)v_3 v_5}, v_6 \mapsto -\frac{v_2^2 - v_3^2}{4(v_2^2 + 1)v_3 v_6}.$$

Define $X_1 = \frac{v_3+\zeta}{v_3-\zeta}$, $X_2 = \frac{v_3v_6}{v_1v_2} \left(\frac{v_2+\zeta}{v_3-\zeta} \right)$, $X_3 = \frac{4v_1v_2}{\eta(v_2-\zeta)} \left(\frac{v_3+\zeta}{v_3-\zeta} \right)$, $X_4 = \left(\frac{v_2+\zeta}{v_2-\zeta} \right) \left(\frac{v_3+\zeta}{v_3-\zeta} \right)$, $X_5 = \left(\frac{v_2+\zeta}{v_2-\zeta} \right) \left(\frac{v_4+1}{v_4-1} \right)$, $X_6 = \left(\frac{v_2+\zeta}{v_2-\zeta} \right) \left(\frac{v_5+\zeta v_6}{v_5-\zeta v_6} \right)$. Then $k(v_1, v_2, v_3, v_4, v_5, v_6) = k(X_1, X_2, X_3, X_4, X_5, X_6)$ and

$$g_1 : X_1 \mapsto -X_1, X_2 \mapsto \frac{X_4}{X_2}, X_3 \mapsto \frac{(X_4-1)(X_4-X_1^2)}{X_3}, X_4 \mapsto X_4, X_5 \mapsto X_5, X_6 \mapsto X_6.$$

The action of g_1 on $k(X_1, X_2, X_3, X_4)$ and that of τ on $k(x_1, x_2, x_3, x_4)$ in 1.24 (ii) are exactly the same. We also have $k(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle g_1 \rangle} = k(X_1, X_2, X_3, X_4)^{\langle g_1 \rangle} (X_5, X_6)$ because g_1 acts on $k(X_5, X_6)$ trivially. Hence $k(V_{242})^G$ is rational over $\varphi(L_k^{(1)})$.

Case 4: $G = G(2^7, 36)$ which belongs to Φ_{39} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = g_4$, $g_2^2 = g_5$, $g_3^2 = 1$, $g_4^2 = 1$, $g_5^2 = 1$, $g_6^2 = 1$, $g_7^2 = 1$, $Z(G) = \langle g_6, g_7 \rangle$, $[g_2, g_1] = g_3$, $[g_3, g_1] = g_6$, $[g_3, g_2] = g_7$, $[g_4, g_2] = g_6$, $[g_5, g_1] = g_7$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{36}) \simeq GL_8(k)$ of dimension 8 which is decomposable into two irreducible components $V_{36} \simeq U_4 \oplus U'_4$ of dimension 4. By Theorem 2.2, $k(G)$ is rational over $k(V_{36})^G$. We will show that $k(V_{36})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{36}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto -y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_6, y_6 \mapsto y_5, y_7 \mapsto y_8, y_8 \mapsto y_7, \\ g_2 : y_1 &\mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto y_4, y_4 \mapsto y_3, y_5 \mapsto -y_7, y_6 \mapsto y_8, y_7 \mapsto y_5, y_8 \mapsto y_6, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_4 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_5 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto y_6, y_7 \mapsto -y_7, y_8 \mapsto y_8, \\ g_6 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_7 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_8}$, $z_5 = \frac{y_6}{y_8}$, $z_6 = \frac{y_7}{y_8}$, $z_7 = y_4$, $z_8 = y_8$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto -\frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{z_5}{z_6}, z_5 \mapsto \frac{z_4}{z_6}, z_6 \mapsto \frac{1}{z_6}, z_7 \mapsto z_2 z_7, z_8 \mapsto z_6 z_8, \\ g_2 : z_1 &\mapsto \frac{z_2}{z_3}, z_2 \mapsto \frac{z_1}{z_3}, z_3 \mapsto \frac{1}{z_3}, z_4 \mapsto -\frac{z_6}{z_5}, z_5 \mapsto \frac{1}{z_5}, z_6 \mapsto \frac{z_4}{z_5}, z_7 \mapsto z_3 z_7, z_8 \mapsto z_5 z_8, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto -z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto -z_7, z_8 \mapsto z_8, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto -z_8. \end{aligned}$$

By applying Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4, z_5, z_6)(z_7, z_8)$, we see that the invariant field $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5, z_6)^G$. Because $Z(G) = \langle g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5, z_6)$ trivially, $k(z_1, z_2, z_3, z_4, z_5, z_6)^G = k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$. Hence it suffices to show that $k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = z_2 z_5$, $u_2 = \frac{z_1}{z_3 z_5}$, $u_3 = -\frac{z_3 z_6}{z_1 z_4}$, $u_4 = \left(\frac{z_2 + z_1 z_3}{z_2 - z_1 z_3} \right) \frac{z_4 z_6}{z_5}$, $u_5 = \frac{z_4 z_6}{z_5}$, $u_6 = \frac{z_1 + z_2 z_3}{z_1 - z_2 z_3}$. Note that $\frac{u'_4 + 1}{u'_4 - 1} = \frac{z_2}{z_1 z_3}$ where $u'_4 = \frac{z_2 + z_1 z_3}{z_2 - z_1 z_3}$ and $\frac{u_6 + 1}{u_6 - 1} = \frac{z_1}{z_2 z_3}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = -8$, $k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_3, g_4, g_5 \rangle} =$

$k(u_1, u_2, u_3, u_4, u_5, u_6)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto \frac{-1}{u_1 u_2 u_3}, u_2 \mapsto u_3, u_3 \mapsto u_2, u_4 \mapsto \frac{1}{u_4}, u_5 \mapsto \frac{1}{u_5}, u_6 \mapsto \frac{-1}{u_6}, \\ g_2 : u_1 &\mapsto u_2, u_2 \mapsto u_1, u_3 \mapsto \frac{-1}{u_1 u_2 u_3}, u_4 \mapsto u_4, u_5 \mapsto -u_5, u_6 \mapsto -u_6. \end{aligned}$$

This action of $\langle g_1, g_2 \rangle$ on $k(u_1, u_2, u_3, u_4, u_5, u_6)$ is exactly the same to the equation (3) in Case 3: $G = G(2^7, 242)$. Hence $k(V_{36})^{G(36)} \simeq k(V_{242})^{G(242)}$ and $k(V_{36})^{G(36)}$ is rational over $\varphi(L_k^{(1)})$.

Case 5: $G = G(2^7, 1924)$ which belongs to Φ_{43} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = 1, g_2^2 = 1, g_3^2 = 1, g_4^2 = 1, g_5^2 = 1, g_6^2 = g_7, g_7^2 = 1, Z(G) = \langle g_5, g_7 \rangle, [g_2, g_1] = g_5, [g_3, g_1] = g_6, [g_3, g_2] = g_5 g_7, [g_4, g_1] = g_5, [g_6, g_1] = g_7, [g_6, g_3] = g_7$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{1924}) \simeq GL_8(k)$ of dimension 8 which is decomposable into two irreducible components $V_{1924} \simeq U_4 \oplus U_4'$ of dimension 4. By Theorem 2.2, $k(G)$ is rational over $k(V_{1924})^G$. We will show that $k(V_{1924})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{1924}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_8, y_6 \mapsto -\zeta y_7, y_7 \mapsto \zeta y_6, y_8 \mapsto y_5, \\ g_2 : y_1 &\mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto -y_4, y_4 \mapsto -y_3, y_5 \mapsto -y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto -y_8, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_7, y_6 \mapsto y_8, y_7 \mapsto y_5, y_8 \mapsto y_6, \\ g_4 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_5 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, \\ g_6 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -\zeta y_5, y_6 \mapsto -\zeta y_6, y_7 \mapsto \zeta y_7, y_8 \mapsto \zeta y_8, \\ g_7 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}, z_2 = \frac{y_2}{y_4}, z_3 = \frac{y_3}{y_4}, z_4 = \frac{y_5}{y_8}, z_5 = \frac{y_6}{y_8}, z_6 = \frac{y_7}{y_8}, z_7 = y_4, z_8 = y_8$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto -\frac{\zeta z_6}{z_4}, z_6 \mapsto \frac{\zeta z_5}{z_4}, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_2 : z_1 &\mapsto -\frac{z_2}{z_3}, z_2 \mapsto -\frac{z_1}{z_3}, z_3 \mapsto \frac{1}{z_3}, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto -z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto \frac{z_6}{z_5}, z_5 \mapsto \frac{1}{z_5}, z_6 \mapsto \frac{z_4}{z_5}, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8. \end{aligned}$$

By applying Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4, z_5, z_6)(z_7, z_8)$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5, z_6)^G$. Because $Z(G) = \langle g_5, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5, z_6)$ trivially, $k(z_1, z_2, z_3, z_4, z_5, z_6)^G = k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_1, g_2, g_3, g_4, g_6 \rangle}$. Hence it suffices to show that $k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_1, g_2, g_3, g_4, g_6 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = \frac{z_1}{z_2}, u_2 = \frac{z_1 z_2}{z_3}, u_3 = z_3, u_4 = \frac{z_4}{z_5}, u_5 = z_5 z_4, u_6 = \frac{z_6 z_4}{z_5}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = 4$,

$k(z_1, z_2, z_3, z_4, z_5, z_6)^{\langle g_4, g_6 \rangle} = k(u_1, u_2, u_3, u_4, u_5, u_6)$ and

$$g_1 : u_1 \mapsto u_3, u_2 \mapsto \frac{1}{u_2}, u_3 \mapsto u_1, u_4 \mapsto -\frac{u_4}{\zeta u_6}, u_5 \mapsto -\frac{\zeta u_6}{u_4^2 u_5}, u_6 \mapsto -\frac{1}{u_6},$$

$$g_2 : u_1 \mapsto \frac{1}{u_1}, u_2 \mapsto u_2, u_3 \mapsto \frac{1}{u_3}, u_4 \mapsto -u_4, u_5 \mapsto -u_5, u_6 \mapsto u_6,$$

$$g_3 : u_1 \mapsto -u_1, u_2 \mapsto u_2, u_3 \mapsto -u_3, u_4 \mapsto \frac{u_6}{u_4}, u_5 \mapsto \frac{u_6}{u_5}, u_6 \mapsto u_6.$$

Define $v_1 = \left(\frac{u_1+1}{u_1-1}\right)\left(\frac{u_3+1}{u_3-1}\right)$, $v_2 = u_2$, $v_3 = \left(\frac{u_1-1}{u_1+1}\right)\left(\frac{u_3-1}{u_3+1}\right)$, $v_4 = u_4\left(\frac{u_1-1}{u_1+1}\right)$, $v_5 = -\frac{\zeta u_6}{u_4}\left(\frac{u_3+1}{u_3-1}\right)$, $v_6 = -\frac{\zeta u_6}{u_4 u_5}$. Then $k(u_1, u_2, u_3, u_4, u_5, u_6)^{\langle g_2 \rangle} = k(v_1, v_2, v_3, v_4, v_5, v_6)$ and

$$g_1 : v_1 \mapsto v_1, v_2 \mapsto \frac{1}{v_2}, v_3 \mapsto \frac{1}{v_3}, v_4 \mapsto \frac{1}{v_5}, v_5 \mapsto \frac{1}{v_4}, v_6 \mapsto \frac{1}{v_6},$$

$$g_3 : v_1 \mapsto \frac{1}{v_1}, v_2 \mapsto v_2, v_3 \mapsto \frac{1}{v_3}, v_4 \mapsto -\frac{v_5}{\zeta v_3}, v_5 \mapsto -\frac{\zeta v_4}{v_3}, v_6 \mapsto -\frac{1}{v_6}.$$

Define $w_1 = \left(\frac{v_1+1}{v_1-1}\right)\left(\frac{v_3+1}{v_3-1}\right)$, $w_2 = -\left(\frac{v_2+1}{v_2-1}\right)\left(\frac{v_6+1}{v_6-1}\right)$, $w_3 = \left(\frac{v_2-1}{v_2+1}\right)\left(\frac{v_6-1}{v_6+1}\right)$, $w_4 = v_4 + \frac{1}{v_5}$, $w_5 = \left(\frac{v_2+1}{v_2-1}\right)\left(v_4 - \frac{1}{v_5}\right)$, $w_6 = \left(\frac{v_2+1}{v_2-1}\right)\left(\frac{v_6+1}{v_6-1}\right)$. Then $k(v_1, v_2, v_3, v_4, v_5, v_6)^{\langle g_1 \rangle} = k(w_1, w_2, w_3, w_4, w_5, w_6)$ and

$$g_3 : w_1 \mapsto w_1, w_2 \mapsto w_6, w_3 \mapsto -w_3, w_4 \mapsto -\frac{4w_2w_6(w_3^2w_4-2w_3w_5-w_2w_4w_6)}{\zeta(w_3^2+w_2w_6)(w_5^2+w_2w_4^2w_6)},$$

$$w_5 \mapsto -\frac{4w_2w_6(w_3^2w_5+2w_2w_3w_4w_6-w_2w_5w_6)}{\zeta(w_3^2+w_2w_6)(w_5^2+w_2w_4^2w_6)}, w_6 \mapsto w_2.$$

Define

$$X_1 = -\frac{w_3w_5+w_2w_4w_6}{w_3(w_3w_4-w_5)}, X_2 = \frac{\zeta(w_3w_5+w_2w_4w_6)}{(w_3w_4-w_5)w_6}, X_3 = -\frac{2\eta(w_3^2w_4-2w_3w_5-w_2w_4w_6)(w_3w_5+w_2w_4w_6)}{w_3w_4w_6(w_3w_4-w_5)^2},$$

$$X_4 = -\frac{(w_3w_5+w_2w_4w_6)^2}{w_2w_6(w_3w_4-w_5)^2}, X_5 = w_1, X_6 = -\frac{w_2(w_3w_4-w_5)w_6}{w_3w_5+w_2w_4w_6}.$$

It follows from $w_1 = X_5$, $w_2 = \zeta X_2 X_6$, $w_3 = -\frac{X_4 X_6}{X_1}$, $w_4 = \frac{2\zeta\eta(X_1+1)X_2}{X_3}$, $w_5 = \frac{2\zeta\eta(X_1+1)(X_1-X_4)X_2X_6}{(X_1-1)X_3}$, $w_6 = \frac{\zeta X_4 X_6}{X_2}$ that $k(w_1, w_2, w_3, w_4, w_5, w_6) = k(X_1, X_2, X_3, X_4, X_5, X_6)$ and

$$g_3 : X_1 \mapsto -X_1, X_2 \mapsto \frac{X_4}{X_2}, X_3 \mapsto \frac{(X_4-1)(X_4-X_1^2)}{X_3}, X_4 \mapsto X_4, X_5 \mapsto X_5, X_6 \mapsto X_6.$$

The action of g_3 on $k(X_1, X_2, X_3, X_4)$ and that of τ on $k(x_1, x_2, x_3, x_4)$ in Definition 1.24 (ii) are exactly the same. Because g_3 acts on $k(X_5, X_6)$ trivially, we have $k(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle g_3 \rangle} = k(X_1, X_2, X_3, X_4)^{\langle g_3 \rangle}(X_5, X_6)$. Hence $k(V_{1924})^G$ is rational over $\varphi(L_k^{(1)})$.

Case 6: $G = G(2^7, 417)$ which belongs to Φ_{58} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = 1$, $g_2^2 = g_4$, $g_3^2 = 1$, $g_4^2 = g_6$, $g_5^2 = g_7$, $g_6^2 = 1$, $g_7^2 = 1$, $Z(G) = \langle g_6, g_7 \rangle$, $[g_2, g_1] = g_4$, $[g_3, g_1] = g_5$, $[g_3, g_2] = g_6$, $[g_4, g_1] = g_6$, $[g_5, g_1] = g_7$, $[g_5, g_3] = g_7$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{417}) \simeq GL_6(k)$ of dimension 6 which is decomposable into two irreducible components $V_{417} \simeq U_4 \oplus U_2$ of dimension 4 and 2 respectively. By Theorem 2.2, $k(G)$ is rational over $k(V_{417})^G$. We will show that $k(V_{417})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{417}) = k(y_1, y_2, y_3, y_4, y_5, y_6)$ is given by

$$g_1 : y_1 \mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto y_4, y_4 \mapsto y_3, y_5 \mapsto \eta^3 y_6, y_6 \mapsto -\eta y_5,$$

$$g_2 : y_1 \mapsto \eta^3 y_1, y_2 \mapsto \eta y_2, y_3 \mapsto -\eta^3 y_3, y_4 \mapsto -\eta y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$g_3 : y_1 \mapsto y_3, y_2 \mapsto -\zeta y_4, y_3 \mapsto y_1, y_4 \mapsto \zeta y_2, y_5 \mapsto y_6, y_6 \mapsto y_5,$$

$$g_4 : y_1 \mapsto -\zeta y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto -\zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$g_5 : y_1 \mapsto -\zeta y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto -\zeta y_4, y_5 \mapsto -\zeta y_5, y_6 \mapsto \zeta y_6,$$

$$g_6 : y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6,$$

$$g_7 : y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6.$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_6}$, $z_5 = y_4$, $z_6 = y_6$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6) = k(z_1, z_2, z_3, z_4, z_5, z_6)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto \frac{z_2}{z_3}, z_2 \mapsto \frac{z_1}{z_3}, z_3 \mapsto \frac{1}{z_3}, z_4 \mapsto -\frac{\zeta}{z_4}, z_5 \mapsto z_3 z_5, z_6 \mapsto -\eta z_4 z_6, \\ g_2 : z_1 &\mapsto -\zeta z_1, z_2 \mapsto -z_2, z_3 \mapsto \zeta z_3, z_4 \mapsto z_4, z_5 \mapsto -\eta z_5, z_6 \mapsto z_6, \\ g_3 : z_1 &\mapsto \frac{z_3}{\zeta z_2}, z_2 \mapsto -\frac{1}{z_2}, z_3 \mapsto \frac{z_1}{\zeta z_2}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto \zeta z_2 z_5, z_6 \mapsto z_4 z_6, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto z_6, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto -z_2, z_3 \mapsto -z_3, z_4 \mapsto -z_4, z_5 \mapsto -\zeta z_5, z_6 \mapsto \zeta z_6, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto -z_6. \end{aligned}$$

Apply Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4)(z_5, z_6)$, the invariant field $k(z_1, z_2, z_3, z_4, z_5, z_6)^G$ is rational over $k(z_1, z_2, z_3, z_4)^G$. We find that $k(z_1, z_2, z_3, z_4)^G = k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ because $Z(G) = \langle g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4)$ trivially. It suffices to show that $k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = \frac{z_1^2 z_3^2}{z_2^2}$, $u_2 = \frac{z_2^2 z_4}{z_1 z_3}$, $u_3 = \frac{\zeta z_1^2}{z_2 z_4}$, $u_4 = \frac{z_1}{z_2 z_3}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = -8$, $k(z_1, z_2, z_3, z_4)^{\langle g_2, g_4, g_5 \rangle} = k(u_1, u_2, u_3, u_4)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto u_3, u_3 \mapsto u_2, u_4 \mapsto \frac{1}{u_4}, \\ g_3 : u_1 &\mapsto u_1, u_2 \mapsto -\frac{1}{u_1 u_2}, u_3 \mapsto -\frac{u_1}{u_3}, u_4 \mapsto -\frac{1}{u_4}. \end{aligned}$$

Hence the action of $\langle g_1, g_3 \rangle$ on $k(u_1, u_2, u_3, u_4)$ and that of $\langle g_1, g_2 \rangle$ on $k(u_1, u_2, u_3, u_4)$ as in (2) of Case 1: $G = G(2^7, 227)$ are exactly the same. Hence $k(V_{417})^{G(417)} \simeq k(V_{227})^{G(227)}$ and $k(V_{417})^{G(417)}$ is rational over $\varphi(L_k^{(1)})$.

Case 7: $G = G(2^7, 446)$ which belongs to Φ_{60} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = 1$, $g_2^2 = g_4$, $g_3^2 = g_5$, $g_4^2 = g_6$, $g_5^2 = g_7$, $g_6^2 = 1$, $g_7^2 = 1$, $Z(G) = \langle g_6, g_7 \rangle$, $[g_2, g_1] = g_4$, $[g_3, g_1] = g_5$, $[g_3, g_2] = g_6$, $[g_4, g_1] = g_6$, $[g_5, g_1] = g_7$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{446}) \simeq GL_6(k)$ of dimension 6 which is decomposable into two irreducible components $V_{446} \simeq U_4 \oplus U_2$ of dimension 4 and 2 respectively. By Theorem 2.2, $k(G)$ is rational over $k(V_{446})^G$. We will show that $k(V_{446})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{446}) = k(y_1, y_2, y_3, y_4, y_5, y_6)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_6, y_6 \mapsto y_5, \\ g_2 : y_1 &\mapsto -\zeta y_2, y_2 \mapsto y_1, y_3 \mapsto -y_4, y_4 \mapsto -\zeta y_3, y_5 \mapsto y_5, y_6 \mapsto y_6, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto w^3 y_5, y_6 \mapsto w y_6, \\ g_4 : y_1 &\mapsto -\zeta y_1, y_2 \mapsto -\zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, \\ g_5 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -\zeta y_5, y_6 \mapsto \zeta y_6, \\ g_6 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, \\ g_7 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_6}$, $z_5 = y_4$, $z_6 = y_6$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6) = k(z_1, z_2, z_3, z_4, z_5, z_6)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto z_2 z_5, z_6 \mapsto z_4 z_6, \\ g_2 : z_1 &\mapsto \frac{z_2}{z_3}, z_2 \mapsto -\frac{z_1}{\zeta z_3}, z_3 \mapsto \frac{1}{\zeta z_3}, z_4 \mapsto z_4, z_5 \mapsto -\zeta z_3 z_5, z_6 \mapsto z_6, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto \zeta z_4, z_5 \mapsto z_5, z_6 \mapsto \eta z_6, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto z_6, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto \zeta z_6, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto -z_6. \end{aligned}$$

Apply Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4)(z_5, z_6)$, the invariant field $k(z_1, z_2, z_3, z_4, z_5, z_6)^G$ is rational over $k(z_1, z_2, z_3, z_4)^G$. We find that $k(z_1, z_2, z_3, z_4)^G = k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ because $Z(G) = \langle g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4)$ trivially. It suffices to show that $k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4, g_5 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = z_4^4$, $u_2 = \frac{\eta z_2}{z_1 z_4^2}$, $u_3 = \frac{\eta z_4^2}{z_3}$, $u_4 = \frac{\zeta z_1 z_2 + z_3}{z_1 z_2 + \zeta z_3}$. Note that $\frac{u_4 + \zeta}{u_4 - \zeta} = \frac{\zeta z_1 z_2}{z_3}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = 8$, $k(z_1, z_2, z_3, z_4)^{\langle g_3, g_4, g_5 \rangle} = k(u_1, u_2, u_3, u_4)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto u_3, u_3 \mapsto u_2, u_4 \mapsto \frac{1}{u_4}, \\ g_2 : u_1 &\mapsto u_1, u_2 \mapsto -\frac{1}{u_1 u_2}, u_3 \mapsto -\frac{u_1}{u_3}, u_4 \mapsto -\frac{1}{u_4}. \end{aligned}$$

Hence the action of $\langle g_1, g_2 \rangle$ on $k(u_1, u_2, u_3, u_4)$ and that of $\langle g_1, g_2 \rangle$ on $k(u_1, u_2, u_3, u_4)$ as in (2) of Case 1: $G = G(2^7, 227)$ are exactly the same. Hence $k(V_{446})^{G(446)} \simeq k(V_{227})^{G(227)}$ and $k(V_{446})^{G(446)}$ is rational over $\varphi(L_k^{(1)})$.

Case 8: $G = G(2^7, 950)$ which belongs to Φ_{80} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = 1$, $g_2^2 = g_4 g_6$, $g_3^2 = 1$, $g_4^2 = g_6 g_7$, $g_5^2 = 1$, $g_6^2 = g_7$, $g_7^2 = 1$, $Z(G) = \langle g_5, g_7 \rangle$, $[g_2, g_1] = g_4$, $[g_3, g_1] = g_5$, $[g_3, g_2] = g_7$, $[g_4, g_1] = g_6$, $[g_6, g_1] = g_7$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{950}) \simeq GL_6(k)$ of dimension 6 which is decomposable into two irreducible components $V_{950} \simeq U_4 \oplus U_2$ of dimension 4 and 2 respectively. By Theorem 2.2, $k(G)$ is rational over $k(V_{950})^G$. We will show that $k(V_{950})^G$ is rational over $\varphi(L_k^{(1)})$.

The action of G on $k(V_{950}) = k(y_1, y_2, y_3, y_4, y_5, y_6)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_6, y_6 \mapsto y_5, \\ g_2 : y_1 &\mapsto \eta^3 y_2, y_2 \mapsto y_1, y_3 \mapsto y_4, y_4 \mapsto -\eta y_3, y_5 \mapsto y_5, y_6 \mapsto y_6, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto y_6, \\ g_4 : y_1 &\mapsto -\eta y_1, y_2 \mapsto -\eta y_2, y_3 \mapsto \eta^3 y_3, y_4 \mapsto \eta^3 y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, \\ g_5 : y_1 &\mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, \\ g_6 : y_1 &\mapsto -\zeta y_1, y_2 \mapsto -\zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, \\ g_7 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_6}$, $z_5 = y_4$, $z_6 = y_6$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6) = k(z_1, z_2, z_3, z_4, z_5, z_6)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto z_2 z_5, z_6 \mapsto z_4 z_6, \\ g_2 : z_1 &\mapsto -\frac{\zeta z_2}{z_3}, z_2 \mapsto -\frac{z_1}{\eta z_3}, z_3 \mapsto -\frac{1}{\eta z_3}, z_4 \mapsto z_4, z_5 \mapsto -\eta z_3 z_5, z_6 \mapsto z_6, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, \\ g_4 : z_1 &\mapsto -\frac{z_1}{\zeta}, z_2 \mapsto -\frac{z_2}{\zeta}, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto \eta^3 z_5, z_6 \mapsto z_6, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto -z_6, \\ g_6 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto z_6, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6. \end{aligned}$$

Apply Theorem 2.3 twice to $k(z_1, z_2, z_3, z_4)(z_5, z_6)$, the invariant field $k(z_1, z_2, z_3, z_4, z_5, z_6)^G$ is rational over $k(z_1, z_2, z_3, z_4)^G$. We find that $k(z_1, z_2, z_3, z_4)^G = k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4, g_6 \rangle}$ because $Z(G) = \langle g_5, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4)$ trivially. It suffices to show that $k(z_1, z_2, z_3, z_4)^{\langle g_1, g_2, g_3, g_4, g_6 \rangle}$ is rational over $\varphi(L_k^{(1)})$.

Define $u_1 = z_4^2$, $u_2 = \frac{\omega z_1}{z_2 z_4}$, $u_3 = \omega z_3 z_4$, $u_4 = \frac{\zeta z_1^2 z_2^2 - z_3^2}{z_1^2 z_2^2 - \zeta z_3^2}$. Note that $\frac{u_4 + \zeta}{u_4 - \zeta} = -\frac{\zeta z_1^2 z_2^2}{z_3^2}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = 8$, $k(z_1, z_2, z_3, z_4)^{\langle g_3, g_4, g_6 \rangle} = k(u_1, u_2, u_3, u_4)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto u_3, u_3 \mapsto u_2, u_4 \mapsto \frac{1}{u_4}, \\ g_2 : u_1 &\mapsto u_1, u_2 \mapsto -\frac{1}{u_1 u_2}, u_3 \mapsto -\frac{u_1}{u_3}, u_4 \mapsto -\frac{1}{u_4}. \end{aligned}$$

Hence the action of $\langle g_1, g_2 \rangle$ on $k(u_1, u_2, u_3, u_4)$ and that of $\langle g_1, g_2 \rangle$ on $k(u_1, u_2, u_3, u_4)$ as in (2) of Case 1: $G = G(2^7, 227)$ are exactly the same. Hence $k(V_{950})^{G(950)} \simeq k(V_{227})^{G(227)}$ and $k(V_{950})^{G(950)}$ is rational over $\varphi(L_k^{(1)})$.

Case 9: $G = G(2^7, 144)$ which belongs to Φ_{106} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = g_4$, $g_2^2 = g_6$, $g_3^2 = g_6 g_7$, $g_4^2 = 1$, $g_5^2 = g_7$, $g_6^2 = 1$, $g_7^2 = 1$, $Z(G) = \langle g_7 \rangle$, $[g_2, g_1] = g_3$, $[g_3, g_1] = g_5$, $[g_3, g_2] = g_6$, $[g_4, g_2] = g_5 g_6$, $[g_4, g_3] = g_6 g_7$, $[g_5, g_1] = g_6$, $[g_5, g_2] = g_7$, $[g_5, g_4] = g_7$, $[g_6, g_1] = g_7$.

Because the center $Z(G)$ of G is cyclic group of order two, there exists a faithful irreducible representation $\rho : G \rightarrow GL(V_{144}) \simeq GL_8(k)$ of dimension 8. By Theorem 2.2, $k(G)$ is rational over $k(V_{144})^G$. We will show that $k(V_{144})^G$ is rational over $\varphi(L_k^{(2)})$.

The action of G on $k(V_{144}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto y_2, y_2 \mapsto y_3, y_3 \mapsto y_6, y_4 \mapsto \zeta y_5, y_5 \mapsto y_8, y_6 \mapsto y_1, y_7 \mapsto y_4, y_8 \mapsto -\zeta y_7, \\ g_2 : y_1 &\mapsto -y_4, y_2 \mapsto \zeta y_5, y_3 \mapsto -y_8, y_4 \mapsto y_1, y_5 \mapsto -\zeta y_2, y_6 \mapsto -\zeta y_7, y_7 \mapsto \zeta y_6, y_8 \mapsto y_3, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto \zeta y_5, y_6 \mapsto -\zeta y_6, y_7 \mapsto -\zeta y_7, y_8 \mapsto y_8, \\ g_4 : y_1 &\mapsto y_3, y_2 \mapsto y_6, y_3 \mapsto y_1, y_4 \mapsto \zeta y_8, y_5 \mapsto -\zeta y_7, y_6 \mapsto y_2, y_7 \mapsto \zeta y_5, y_8 \mapsto -\zeta y_4, \\ g_5 : y_1 &\mapsto -\zeta y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto -\zeta y_5, y_6 \mapsto -\zeta y_6, y_7 \mapsto \zeta y_7, y_8 \mapsto -\zeta y_8, \\ g_6 : y_1 &\mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto -y_8, \\ g_7 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_8}$, $z_2 = \frac{y_2}{y_8}$, $z_3 = \frac{y_3}{y_8}$, $z_4 = \frac{y_4}{y_8}$, $z_5 = \frac{y_5}{y_8}$, $z_6 = \frac{y_6}{y_8}$, $z_7 = \frac{y_7}{y_8}$. Then we obtain that $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto -\frac{z_2}{z_7}, z_2 \mapsto -\frac{z_3}{z_7}, z_3 \mapsto -\frac{z_6}{z_7}, z_4 \mapsto -\frac{z_5}{z_7}, z_5 \mapsto -\frac{1}{z_7}, z_6 \mapsto -\frac{z_1}{z_7}, z_7 \mapsto -\frac{z_4}{z_7}, z_8 \mapsto -\zeta z_7 z_8, \\ g_2 : z_1 &\mapsto -\frac{z_4}{z_3}, z_2 \mapsto \frac{\zeta z_5}{z_3}, z_3 \mapsto -\frac{1}{z_3}, z_4 \mapsto \frac{z_1}{z_3}, z_5 \mapsto -\frac{\zeta z_2}{z_3}, z_6 \mapsto -\frac{\zeta z_7}{z_3}, z_7 \mapsto \frac{\zeta z_6}{z_3}, z_8 \mapsto z_3 z_8, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto \zeta z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto -\zeta z_6, z_7 \mapsto -\zeta z_7, z_8 \mapsto z_8, \\ g_4 : z_1 &\mapsto -\frac{z_3}{z_4}, z_2 \mapsto -\frac{z_6}{z_4}, z_3 \mapsto -\frac{z_1}{z_4}, z_4 \mapsto -\frac{1}{z_4}, z_5 \mapsto \frac{z_7}{z_4}, z_6 \mapsto -\frac{z_2}{z_4}, z_7 \mapsto -\frac{z_5}{z_4}, z_8 \mapsto -\zeta z_4 z_8, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto -z_2, z_3 \mapsto -z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto -z_7, z_8 \mapsto -\zeta z_8, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto -z_6, z_7 \mapsto -z_7, z_8 \mapsto -z_8, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto -z_8. \end{aligned}$$

Apply Theorem 2.3 to $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)(z_8)$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^G$. Because $Z(G) = \langle g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ trivially, we have $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^G = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$. It suffices to show that $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ is rational over $\varphi(L_k^{(2)})$.

Define $u_1 = -\frac{z_2 z_7}{z_5 z_6}$, $u_2 = -\frac{z_3}{z_2 z_5}$, $u_3 = \frac{z_3 z_4}{z_1}$, $u_4 = -\frac{z_7}{z_4 z_6}$, $u_5 = \frac{z_5}{z_3 z_7}$, $u_6 = \frac{z_2 z_3}{z_6}$, $u_7 = \frac{z_3 z_6}{z_1 z_7}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = -8$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_3, g_5, g_6 \rangle} = k(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto -u_3, u_2 \mapsto -\frac{\zeta}{u_2 u_5 u_6}, u_3 \mapsto \frac{1}{u_1}, u_4 \mapsto -\frac{u_1 u_3}{u_6}, u_5 \mapsto \zeta u_4, u_6 \mapsto -\frac{u_7}{\zeta}, u_7 \mapsto -\frac{1}{u_1 u_3 u_5}, \\ g_2 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto -u_2, u_3 \mapsto \frac{1}{u_3}, u_4 \mapsto u_7, u_5 \mapsto u_6, u_6 \mapsto u_5, u_7 \mapsto u_4, \\ g_4 : u_1 &\mapsto -\frac{1}{u_1}, u_2 \mapsto -\frac{u_2 u_5 u_6}{u_4 u_7}, u_3 \mapsto -\frac{1}{u_3}, u_4 \mapsto -\frac{\zeta u_3}{u_1 u_7}, u_5 \mapsto -\frac{\zeta u_1 u_3}{u_6}, u_6 \mapsto \frac{1}{\zeta u_1 u_3 u_5}, u_7 \mapsto \frac{u_1}{\zeta u_3 u_4}. \end{aligned}$$

Note that $g_1^2 = g_4$. Thus we will omit the presentation of the action g_4 . Define $v_1 = -\left(\frac{u_1+1}{u_1-1}\right)/\left(\frac{u_3+1}{u_3-1}\right)$, $v_2 = u_2\left(\frac{u_1+1}{u_1-1}\right)$, $v_3 = \left(\frac{u_3+1}{u_3-1}\right)\left(\frac{u_1+1}{u_1-1}\right)$, $v_4 = u_4 + u_7$, $v_5 = u_5 + u_6$, $v_6 = \left(\frac{u_4-u_7}{u_4+u_7}\right)\left(\frac{u_1+1}{u_1-1}\right)$, $v_7 = \left(\frac{u_5-u_6}{u_5+u_6}\right)\left(\frac{u_1+1}{u_1-1}\right)$. Then $k(u_1, u_2, u_3, u_4, u_5, u_6, u_7)^{\langle g_2 \rangle} = k(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and

$$\begin{aligned} g_1 : v_1 &\mapsto \frac{1}{v_3}, v_2 \mapsto \frac{4\zeta v_1^2 v_3}{v_2 v_5^2 (v_1 v_3 + v_7^2)}, v_3 \mapsto v_1, v_4 \mapsto -\frac{4v_1 v_3 (v_1 - v_3 + 4v_1 v_3 - v_1^2 v_3 + v_1 v_3^2 - 2v_7 + 2v_1 v_7 - 2v_3 v_7 + 2v_1 v_3 v_7)}{(v_1 + v_3)(v_1 v_3 + 1)v_5 (v_1 v_3 + v_7^2)}, \\ v_5 &\mapsto \zeta v_4, v_6 \mapsto \frac{2v_1 v_3 - 2v_1^2 v_3 + 2v_1 v_3^2 - 2v_1^2 v_3^2 + v_1 v_7 - v_3 v_7 + 4v_1 v_3 v_7 - v_1^2 v_3 v_7 + v_1 v_3^2 v_7}{v_3 (v_1 - v_3 + 4v_1 v_3 - v_1^2 v_3 + v_1 v_3^2 - 2v_7 + 2v_1 v_7 - 2v_3 v_7 + 2v_1 v_3 v_7)}, v_7 \mapsto \frac{v_6}{v_3}. \end{aligned}$$

Define

$$\begin{aligned} w_1 &= \frac{v_1+1}{v_1-1}, w_2 = \frac{v_2 v_5}{v_4}, w_3 = \frac{v_3+1}{v_3-1}, w_4 = v_4, w_5 = v_5, \\ w_6 &= -\frac{v_3 + 2v_1 v_3 + v_1^2 v_3 + v_6 + v_1 v_6 - v_3 v_6 - v_1 v_3 v_6}{(v_3-1)(-v_1 + v_1 v_3 + v_6 + v_1 v_6)}, w_7 = \frac{v_1 + 2v_1 v_3 + v_1 v_3^2 - v_7 + v_1 v_7 - v_3 v_7 + v_1 v_3 v_7}{(v_1-1)(-v_3 + v_1 v_3 - v_7 - v_3 v_7)}. \end{aligned}$$

Then it follows from

$$v_1 = \frac{w_1+1}{w_1-1}, v_2 = \frac{w_2 w_4}{w_5}, v_3 = \frac{w_3+1}{w_3-1}, v_4 = w_4, v_5 = w_5, v_6 = \frac{w_6 - w_1^2 (w_3^2 + w_6 - 1)}{w_1 (w_1-1)(w_3-1)(w_6-1)}, v_7 = -\frac{w_7 + w_3^2 (w_1^2 - w_7 - 1)}{w_3 (w_1-1)(w_3-1)(w_7+1)}$$

that $k(v_1, v_2, v_3, v_4, v_5, v_6, v_7) = k(w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ and

$$\begin{aligned} g_1 : w_1 &\mapsto -w_3, w_2 \mapsto \frac{(w_1+1)(w_7+1)}{(w_1-1)w_2(w_7-1)}, w_3 \mapsto w_1, \\ w_4 &\mapsto -\frac{4(w_1^2-1)(w_3^2-1)w_3^2(w_7^2-1)}{(w_1^2 w_3^2-1)w_5(-w_3^2+w_1^2 w_3^2-w_7^2+w_3^2 w_7^2)}, w_5 \mapsto \zeta w_4, w_6 \mapsto w_7, w_7 \mapsto -w_6. \end{aligned}$$

We also define

$$\begin{aligned} p_1 &= w_1, p_2 = -w_3, p_3 = w_6, p_4 = w_7, p_5 = (w_1-1)w_2(w_7-1), \\ p_6 &= -\frac{(w_1^2 w_3^2-1)w_5(w_3^2-w_1^2 w_3^2+w_7^2-w_3^2 w_7^2)}{(w_1-1)w_3}, p_7 = \frac{(w_1^2 w_3^2-1)w_4(w_1^2-w_1^2 w_3^2+w_6^2-w_1^2 w_6^2)\zeta}{w_1(w_3+1)}. \end{aligned}$$

Then it follows from

$$\begin{aligned} w_1 &= p_1, w_2 = \frac{p_5}{(p_1-1)(p_4-1)}, w_3 = -p_2, w_4 = \frac{p_1(p_2-1)p_7}{\zeta(p_1^2p_2^2-1)(p_1^2(p_2^2+p_3^2-1)-p_3^2)}, \\ w_5 &= -\frac{(p_1-1)p_2p_6}{(p_1^2p_2^2-1)(p_2^2(p_1^2+p_4^2-1)-p_4^2)}, w_6 = p_3, w_7 = p_4 \end{aligned}$$

that $k(w_1, w_2, w_3, w_4, w_5, w_6, w_7) = k(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ and

$$\begin{aligned} g_1 : p_1 &\mapsto p_2, p_2 \mapsto -p_1, p_3 \mapsto p_4, p_4 \mapsto -p_3, p_5 \mapsto -\frac{(p_1+1)(p_2-1)(p_3+1)(p_4+1)}{p_5}, \\ p_6 &\mapsto p_7, p_7 \mapsto -\frac{4\zeta(p_2^2-1)(p_1^2p_2^2-1)(p_4^2-1)(p_1^2p_2^2-p_2^2+p_2^2p_4^2-p_4^2)}{p_6}. \end{aligned}$$

Define

$$\begin{aligned} X_1 &= p_2, X_2 = -p_1, X_3 = \frac{p_4}{p_2}, X_4 = \frac{p_3}{p_1}, X_5 = \frac{p_6}{2\eta(p_2+1)(p_4+1)p_2}, \\ X_6 &= -\frac{p_7}{2\eta(p_1-1)(p_3-1)p_1}, X_7 = \frac{p_1p_2(p_2-p_3+p_2p_3+p_5-1)}{p_2-p_3+p_2p_3-p_5-1}. \end{aligned}$$

Then it follows from

$$\begin{aligned} p_1 &= -X_2, p_2 = X_1, p_3 = -X_2X_4, p_4 = X_1X_3, p_5 = \frac{(X_1-1)(X_2X_4-1)(X_1X_2+X_7)}{X_1X_2-X_7}, \\ p_6 &= 2\eta X_1(X_1+1)(X_1X_3+1)X_5, p_7 = 2\eta X_2(X_2+1)(X_2X_4+1)X_6 \end{aligned}$$

that $k(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$. We see that the action of $\langle g_1 \rangle$ on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ and that of $\langle \rho \rangle$ in Definition 1.30 (i) are exactly the same. We also have $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)^{\langle g_1 \rangle} = k(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle g_1 \rangle}(X_7)$ because X_7 is an invariant under the action of $\langle g_1 \rangle$. Hence $k(V_{144})^G$ is rational over $\varphi(L_k^{(2)})$.

Case 10: $G = G(2^7, 138)$ which belongs to Φ_{114} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = g_4, g_2^2 = 1, g_3^2 = g_6, g_4^2 = 1, g_5^2 = g_7, g_6^2 = 1, g_7^2 = 1, Z(G) = \langle g_7 \rangle, [g_2, g_1] = g_3, [g_3, g_1] = g_5, [g_3, g_2] = g_6, [g_4, g_2] = g_5g_6g_7, [g_4, g_3] = g_6g_7, [g_5, g_1] = g_6, [g_5, g_2] = g_7, [g_5, g_4] = g_7, [g_6, g_1] = g_7$.

Because the center $Z(G)$ of G is cyclic group of order two, there exists a faithful irreducible representation $\rho : G \rightarrow GL(V_{138}) \simeq GL_8(k)$ of dimension 8. By Theorem 2.2, $k(G)$ is rational over $k(V_{138})^G$. We will show that $k(V_{138})^G$ is rational over $\varphi(L_k^{(2)})$.

The action of G on $k(V_{138}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ is given by

$$\begin{aligned} g_1 : y_1 &\mapsto y_5, y_2 \mapsto y_1, y_3 \mapsto \zeta y_7, y_4 \mapsto y_2, y_5 \mapsto y_4, y_6 \mapsto y_3, y_7 \mapsto y_8, y_8 \mapsto -\zeta y_6, \\ g_2 : y_1 &\mapsto y_3, y_2 \mapsto -\zeta y_6, y_3 \mapsto y_1, y_4 \mapsto y_8, y_5 \mapsto -\zeta y_7, y_6 \mapsto \zeta y_2, y_7 \mapsto \zeta y_5, y_8 \mapsto y_4, \\ g_3 : y_1 &\mapsto -y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto \zeta y_5, y_6 \mapsto -\zeta y_6, y_7 \mapsto -\zeta y_7, y_8 \mapsto y_8, \\ g_4 : y_1 &\mapsto y_4, y_2 \mapsto y_5, y_3 \mapsto \zeta y_8, y_4 \mapsto y_1, y_5 \mapsto y_2, y_6 \mapsto \zeta y_7, y_7 \mapsto -\zeta y_6, y_8 \mapsto -\zeta y_3, \\ g_5 : y_1 &\mapsto -\zeta y_1, y_2 \mapsto \zeta y_2, y_3 \mapsto \zeta y_3, y_4 \mapsto \zeta y_4, y_5 \mapsto -\zeta y_5, y_6 \mapsto -\zeta y_6, y_7 \mapsto \zeta y_7, y_8 \mapsto -\zeta y_8, \\ g_6 : y_1 &\mapsto y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto -y_7, y_8 \mapsto y_8, \\ g_7 : y_1 &\mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8. \end{aligned}$$

Define $z_1 = \frac{y_1}{y_8}$, $z_2 = \frac{y_2}{y_8}$, $z_3 = \frac{y_3}{y_8}$, $z_4 = \frac{y_4}{y_8}$, $z_5 = \frac{y_5}{y_8}$, $z_6 = \frac{y_6}{y_8}$, $z_7 = \frac{y_7}{y_8}$. Then we obtain that $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)$ and

$$\begin{aligned} g_1 : z_1 &\mapsto -\frac{z_5}{z_6}, z_2 \mapsto -\frac{z_1}{z_6}, z_3 \mapsto -\frac{z_7}{z_6}, z_4 \mapsto -\frac{z_2}{z_6}, z_5 \mapsto -\frac{z_4}{z_6}, z_6 \mapsto -\frac{z_3}{z_6}, z_7 \mapsto -\frac{1}{z_6}, z_8 \mapsto -\zeta z_6 z_8, \\ g_2 : z_1 &\mapsto \frac{z_3}{z_4}, z_2 \mapsto -\frac{\zeta z_6}{z_4}, z_3 \mapsto \frac{z_1}{z_4}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto -\frac{\zeta z_7}{z_4}, z_6 \mapsto \frac{\zeta z_2}{z_4}, z_7 \mapsto \frac{\zeta z_5}{z_4}, z_8 \mapsto z_4 z_8, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto \zeta z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto \zeta z_5, z_6 \mapsto -\zeta z_6, z_7 \mapsto -\zeta z_7, z_8 \mapsto z_8, \\ g_4 : z_1 &\mapsto -\frac{z_4}{z_3}, z_2 \mapsto -\frac{z_5}{z_3}, z_3 \mapsto -\frac{1}{z_3}, z_4 \mapsto -\frac{z_1}{z_3}, z_5 \mapsto -\frac{z_2}{z_3}, z_6 \mapsto -\frac{z_7}{z_3}, z_7 \mapsto \frac{z_6}{z_3}, z_8 \mapsto -\zeta z_3 z_8, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto -z_2, z_3 \mapsto -z_3, z_4 \mapsto -z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto -z_7, z_8 \mapsto -\zeta z_8, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto -z_6, z_7 \mapsto -z_7, z_8 \mapsto z_8, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto -z_8. \end{aligned}$$

Apply Theorem 2.3 to $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)(z_8)$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^G$. Because $Z(G) = \langle g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ trivially, we have $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^G = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$. It suffices to show that $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle}$ is rational over $\varphi(L_k^{(2)})$.

Define $u_1 = -\frac{z_2 z_7}{z_5 z_6}$, $u_2 = -\frac{z_6 z_7}{z_3 z_4}$, $u_3 = \frac{z_3 z_4}{z_1}$, $u_4 = -\frac{z_4 z_6}{z_7}$, $u_5 = \frac{z_5}{z_3 z_7}$, $u_6 = -\frac{\zeta z_2 z_3}{z_6}$, $u_7 = -\frac{z_1 z_7}{z_3 z_6}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = 8$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_3, g_5, g_6 \rangle} = k(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto -\frac{1}{u_3}, u_2 \mapsto -\frac{1}{u_2}, u_3 \mapsto u_1, u_4 \mapsto u_6, u_5 \mapsto u_4, u_6 \mapsto u_7, u_7 \mapsto u_5, \\ g_2 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto -\frac{u_2 u_5 u_6}{u_4 u_7}, u_3 \mapsto \frac{1}{u_3}, u_4 \mapsto -\frac{u_1}{\zeta u_3 u_7}, u_5 \mapsto -\frac{\zeta u_1 u_3}{u_6}, u_6 \mapsto -\frac{\zeta}{u_1 u_3 u_5}, u_7 \mapsto -\frac{u_3}{\zeta u_1 u_4}, \\ g_4 : u_1 &\mapsto -\frac{1}{u_1}, u_2 \mapsto u_2, u_3 \mapsto -\frac{1}{u_3}, u_4 \mapsto u_7, u_5 \mapsto u_6, u_6 \mapsto u_5, u_7 \mapsto u_4. \end{aligned}$$

Note that $g_1^2 = g_4$. Thus we will omit the presentation of the action g_4 . Define $v_1 = \frac{u_1 + \zeta}{u_1 - \zeta}$, $v_2 = \frac{u_2 u_5 u_6 + u_4 u_7 \zeta}{u_2 - \zeta}$, $v_3 = -\frac{u_3 + \zeta}{u_3 - \zeta}$, $v_4 = u_4$, $v_5 = u_6$, $v_6 = \frac{\zeta u_1}{u_3 u_7}$, $v_7 = -\frac{\zeta}{u_1 u_3 u_5}$. Then $k(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = k(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and

$$\begin{aligned} g_1 : v_1 &\mapsto v_3, v_2 \mapsto -v_2, v_3 \mapsto -v_1, \\ v_4 &\mapsto v_5, v_5 \mapsto \frac{(v_1 + 1)(v_3 + 1)\zeta}{(v_1 - 1)(v_3 - 1)v_6}, v_6 \mapsto v_7, v_7 \mapsto \frac{(v_1 - 1)(v_3 - 1)\zeta}{(v_1 + 1)(v_3 + 1)v_4}, \\ g_2 : v_1 &\mapsto -\frac{1}{v_1}, v_2 \mapsto -\frac{1}{v_2}, v_3 \mapsto -\frac{1}{v_3}, v_4 \mapsto v_6, v_5 \mapsto v_7, v_6 \mapsto v_4, v_7 \mapsto v_5. \end{aligned}$$

Define $w_1 = \left(\frac{v_1 + \zeta}{v_1 - \zeta}\right)^2$, $w_2 = \left(\frac{v_1 + \zeta}{v_1 - \zeta}\right)\left(\frac{v_2 + \zeta}{v_2 - \zeta}\right)$, $w_3 = \left(\frac{v_1 + \zeta}{v_1 - \zeta}\right)\left(\frac{v_3 + \zeta}{v_3 - \zeta}\right)$, $w_4 = v_4 + v_6$, $w_5 = v_5 + v_7$, $w_6 = \left(\frac{v_1 + \zeta}{v_1 - \zeta}\right)(v_4 - v_6)$, $w_7 = \left(\frac{v_1 + \zeta}{v_1 - \zeta}\right)(v_5 - v_7)$. Then $k(v_1, v_2, v_3, v_4, v_5, v_6, v_7)^{\langle g_2 \rangle} = k(w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ and

$$\begin{aligned} g_1 : w_1 &\mapsto \frac{w_3^2}{w_1}, w_2 \mapsto \frac{w_3}{w_2}, w_3 \mapsto \frac{w_3}{w_1}, w_4 \mapsto w_5, w_5 \mapsto \frac{4w_1(2(w_3 - 1)(w_1 + w_3)w_6 - \zeta(w_1 - w_1^2 - 4w_1 w_3 - w_3^2 + w_1 w_3^2)w_4)}{(w_1 + 1)(w_1 + w_3^2)(w_1 w_4^2 - w_6^2)}, \\ w_6 &\mapsto \frac{w_3 w_7}{w_1}, w_7 \mapsto \frac{4w_3(2w_1(w_3 - 1)(w_1 + w_3)w_4 - \zeta(w_1 - w_1^2 - 4w_1 w_3 - w_3^2 + w_1 w_3^2)w_6)}{(w_1 + 1)(w_1 + w_3^2)(w_1 w_4^2 - w_6^2)}. \end{aligned}$$

Define

$$\begin{aligned} p_1 &= \frac{w_1 + w_3}{w_1 - w_3}, p_2 = \frac{2w_2}{w_3 - 1}, p_3 = \frac{w_3 + 1}{w_3 - 1}, p_4 = \frac{2w_1(w_1 + w_3)(w_3 - 1)w_4}{(w_1 + 1)(w_1 + w_3^2)w_6} - \frac{(w_1 - w_1^2 - w_3^2 + w_1 w_3^2 - 4w_1 w_3)\zeta}{(w_1 + 1)(w_1 + w_3^2)}, \\ p_5 &= -\frac{2w_1(w_1 - w_3)(w_3 + 1)w_5}{(w_1 + 1)(w_1 + w_3^2)w_7} + \frac{(w_1 - w_1^2 - w_3^2 + w_1 w_3^2 + 4w_1 w_3)\zeta}{(w_1 + 1)(w_1 + w_3^2)}, p_6 = w_6, p_7 = \frac{w_3 w_7}{w_1}. \end{aligned}$$

Then it follows from

$$\begin{aligned} w_1 &= \left(\frac{p_1 + 1}{p_1 - 1}\right)\left(\frac{p_3 + 1}{p_3 - 1}\right), w_2 = \frac{p_2}{p_3 - 1}, w_3 = \frac{p_3 + 1}{p_3 - 1}, w_4 = \frac{(p_1^2 p_3^2 - 1)p_4 + \zeta(2p_1^2 - p_1^2 p_3^2 - 1)}{2p_1(p_1 + 1)(p_3 + 1)}p_6, \\ w_5 &= -\frac{(p_1^2 p_3^2 - 1)p_5 + \zeta(2p_3^2 - p_1^2 p_3^2 - 1)}{2p_3(p_1 - 1)(p_3 + 1)}p_7, w_6 = p_6, w_7 = \left(\frac{p_1 + 1}{p_1 - 1}\right)p_7 \end{aligned}$$

that $k(w_1, w_2, w_3, w_4, w_5, w_6, w_7) = k(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ and

$$g_1 : p_1 \mapsto p_3, p_2 \mapsto -\frac{(p_1+1)(p_3+1)}{p_2}, p_3 \mapsto -p_1, p_4 \mapsto p_5, p_5 \mapsto -\frac{1}{p_4}, p_6 \mapsto p_7, \\ p_7 \mapsto \frac{16p_1^2(p_1^2-1)(p_3^2-1)p_4((p_1^2p_3^2-1)(p_4^2-1)-2(2p_1^2-p_1^2p_3^2-1)p_4\zeta)}{(p_1^2p_3^2-1)(2p_4^2(-8p_1^2+8p_1^4+6p_1^2p_3^2-8p_1^4p_3^2+p_1^4p_3^4+1)+(p_1^2p_3^2-1)^2(p_4^4+1))p_6}.$$

We also define

$$X_1 = p_1, X_2 = p_3, X_3 = \frac{p_4^2+2p_4\zeta-1}{p_1(p_4^2+1)}, X_4 = \frac{p_5^2+2p_5\zeta-1}{p_3(p_5^2+1)}, \\ X_5 = \frac{(p_1^2p_3^2-1)p_4^3p_6(p_4(4p_1^2-p_1^2p_3^2+p_4^2-p_1^2p_3^2p_4^2-3)+(p_1^2p_3^2+3p_4^2-4p_1^2p_4^2+p_1^2p_3^2p_4^2-1)\zeta)}{4\eta p_1^2(p_1+1)(p_3+1)(p_4^2+1)}, \\ X_6 = -\frac{(p_1^2p_3^2-1)p_5^3p_7(p_5(4p_3^2-p_1^2p_3^2+p_5^2-p_1^2p_3^2p_5^2-3)+(p_1^2p_3^2+3p_5^2-4p_3^2p_5^2+p_1^2p_3^2p_5^2-1)\zeta)}{4\eta p_3^2(p_1-1)(p_3+1)(p_5^2+1)}, \\ X_7 = \frac{p_1p_3((p_1+p_2+1)(p_1-p_2+1)-2\zeta(p_1+1)p_2)}{p_1^2+2p_1+p_2^2+1}.$$

Then it follows from

$$p_1 = X_1, p_2 = \frac{(X_1+1)(X_7-X_1X_2)\zeta}{X_7+X_1X_2}, p_3 = X_2, p_4 = \frac{X_1X_3+1}{X_1X_3-1}\zeta, p_5 = \frac{X_2X_4+1}{X_2X_4-1}\zeta, \\ p_6 = -\frac{2\eta(X_1+1)(X_2+1)(X_1X_3-1)^4X_5}{(X_1^2X_2^2-1)(X_1X_3+1)^3(X_2^2-X_3^2+X_1^2X_3^2-1)}, p_7 = \frac{2\eta(X_1-1)(X_2+1)(X_2X_4-1)^4X_6}{(X_1^2X_2^2-1)(X_2X_4+1)^3(X_1^2-X_4^2+X_2^2X_4^2-1)}$$

that $k(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$. We see that the action of $\langle g_1 \rangle$ on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ and that of $\langle \rho \rangle$ in Definition 1.30 (i) are exactly the same. We also have $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)^{\langle g_1 \rangle} = k(X_1, X_2, X_3, X_4, X_5, X_6)^{\langle g_1 \rangle}(X_7)$ because X_7 is an invariant under the action of $\langle g_1 \rangle$. Hence $k(V_{138})^G$ is rational over $\varphi(L_k^{(2)})$.

Case 11: $G = G(2^7, 1544)$ which belongs to Φ_{30} .

$G = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7 \rangle$ with relations $g_1^2 = 1, g_2^2 = 1, g_3^2 = 1, g_4^2 = 1, g_5^2 = 1, g_6^2 = 1, g_7^2 = 1$, $Z(G) = \langle g_5, g_6, g_7 \rangle$, $[g_2, g_1] = g_5$, $[g_3, g_1] = g_6$, $[g_3, g_2] = g_7$, $[g_4, g_2] = g_5g_6$, $[g_4, g_3] = g_5$.

There exists a faithful representation $\rho : G \rightarrow GL(V_{1544}) \simeq GL_{10}(k)$ of dimension 10 which is decomposable into three irreducible components $V_{1544} \simeq U_4 \oplus U_2 \oplus U'_4$ of dimension 4, 2 and 4 respectively. By Theorem 2.2, $k(G)$ is rational over $k(V_{1544})^G$. We will show that $k(V_{1544})^G$ is rational over $\varphi(L_k^{(3)})$.

The action of G on $k(V_{1544}) = k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10})$ is given by

$$g_1 : y_1 \mapsto y_3, y_2 \mapsto y_4, y_3 \mapsto y_1, y_4 \mapsto y_2, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_9, y_8 \mapsto y_{10}, y_9 \mapsto y_7, y_{10} \mapsto y_8, \\ g_2 : y_1 \mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto y_4, y_4 \mapsto y_3, y_5 \mapsto y_6, y_6 \mapsto y_5, y_7 \mapsto -y_8, y_8 \mapsto -y_7, y_9 \mapsto y_{10}, y_{10} \mapsto y_9, \\ g_3 : y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto y_6, y_7 \mapsto y_8, y_8 \mapsto y_7, y_9 \mapsto y_{10}, y_{10} \mapsto y_9, \\ g_4 : y_1 \mapsto -y_1, y_2 \mapsto y_2, y_3 \mapsto -y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto -y_7, y_8 \mapsto y_8, y_9 \mapsto -y_9, y_{10} \mapsto y_{10}, \\ g_5 : y_1 \mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto -y_7, y_8 \mapsto -y_8, y_9 \mapsto -y_9, y_{10} \mapsto -y_{10}, \\ g_6 : y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto -y_3, y_4 \mapsto -y_4, y_5 \mapsto y_5, y_6 \mapsto y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, y_9 \mapsto y_9, y_{10} \mapsto y_{10}, \\ g_7 : y_1 \mapsto y_1, y_2 \mapsto y_2, y_3 \mapsto y_3, y_4 \mapsto y_4, y_5 \mapsto -y_5, y_6 \mapsto -y_6, y_7 \mapsto y_7, y_8 \mapsto y_8, y_9 \mapsto y_9, y_{10} \mapsto y_{10}.$$

Define $z_1 = \frac{y_1}{y_4}$, $z_2 = \frac{y_2}{y_4}$, $z_3 = \frac{y_3}{y_4}$, $z_4 = \frac{y_5}{y_6}$, $z_5 = \frac{y_7}{y_{10}}$, $z_6 = \frac{y_8}{y_{10}}$, $z_7 = \frac{y_9}{y_{10}}$, $z_8 = y_4$, $z_9 = y_6$, $z_{10} = y_{10}$. Then $k(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}) = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10})$ and

$$\begin{aligned} g_1 : z_1 &\mapsto \frac{z_3}{z_2}, z_2 \mapsto \frac{1}{z_2}, z_3 \mapsto \frac{z_1}{z_2}, z_4 \mapsto z_4, z_5 \mapsto \frac{z_7}{z_6}, z_6 \mapsto \frac{1}{z_6}, z_7 \mapsto \frac{z_5}{z_6}, z_8 \mapsto z_2 z_8, z_9 \mapsto z_9, z_{10} \mapsto z_{10} z_6, \\ g_2 : z_1 &\mapsto \frac{z_2}{z_3}, z_2 \mapsto \frac{z_1}{z_3}, z_3 \mapsto \frac{1}{z_3}, z_4 \mapsto \frac{1}{z_4}, z_5 \mapsto -\frac{z_6}{z_7}, z_6 \mapsto -\frac{z_5}{z_7}, z_7 \mapsto \frac{1}{z_7}, z_8 \mapsto z_3 z_8, z_9 \mapsto z_4 z_9, z_{10} \mapsto z_{10} z_7, \\ g_3 : z_1 &\mapsto -z_1, z_2 \mapsto -z_2, z_3 \mapsto z_3, z_4 \mapsto -z_4, z_5 \mapsto \frac{z_6}{z_7}, z_6 \mapsto \frac{z_5}{z_7}, z_7 \mapsto \frac{1}{z_7}, z_8 \mapsto z_8, z_9 \mapsto z_9, z_{10} \mapsto z_{10} z_7, \\ g_4 : z_1 &\mapsto -z_1, z_2 \mapsto z_2, z_3 \mapsto -z_3, z_4 \mapsto z_4, z_5 \mapsto -z_5, z_6 \mapsto z_6, z_7 \mapsto -z_7, z_8 \mapsto z_8, z_9 \mapsto z_9, z_{10} \mapsto z_{10}, \\ g_5 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, z_9 \mapsto z_9, z_{10} \mapsto -z_{10}, \\ g_6 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto -z_8, z_9 \mapsto z_9, z_{10} \mapsto z_{10}, \\ g_7 : z_1 &\mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3, z_4 \mapsto z_4, z_5 \mapsto z_5, z_6 \mapsto z_6, z_7 \mapsto z_7, z_8 \mapsto z_8, z_9 \mapsto -z_9, z_{10} \mapsto z_{10}. \end{aligned}$$

By applying Theorem 2.3 three times to $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)(z_8, z_9, z_{10})$, we obtain that $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10})^G$ is rational over $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^G$. We also find that $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^G = k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_1, g_2, g_3, g_4 \rangle}$ because $Z(G) = \langle g_5, g_6, g_7 \rangle$ acts on $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ trivially. Hence it suffices to show that $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_1, g_2, g_3, g_4 \rangle}$ is rational over $\varphi(L_k^{(3)})$.

Define $u_1 = \frac{z_1}{z_2 z_7}$, $u_2 = \frac{z_1}{z_3}$, $u_3 = \frac{z_1 z_3}{z_2}$, $u_4 = z_4$, $u_5 = \frac{z_1 z_5}{z_2}$, $u_6 = \frac{z_1 z_6}{z_2 z_7}$, $u_7 = \frac{z_1 z_7}{z_2}$. By evaluating the determinant of the matrix M of exponents as in Case 1 (see the equation (1)), we have $\det M = 2$, $k(z_1, z_2, z_3, z_4, z_5, z_6, z_7)^{\langle g_4 \rangle} = k(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ and

$$\begin{aligned} g_1 : u_1 &\mapsto \frac{u_3 u_6}{u_1 u_5}, u_2 \mapsto \frac{1}{u_2}, u_3 \mapsto u_3, u_4 \mapsto u_4, u_5 \mapsto \frac{u_3}{u_6}, u_6 \mapsto \frac{u_3}{u_5}, u_7 \mapsto \frac{u_3 u_5}{u_6 u_7}, \\ g_2 : u_1 &\mapsto \frac{1}{u_1}, u_2 \mapsto \frac{u_2 u_3}{u_1 u_7}, u_3 \mapsto \frac{1}{u_3}, u_4 \mapsto \frac{1}{u_4}, u_5 \mapsto -\frac{u_6}{u_1 u_7}, u_6 \mapsto -\frac{u_5}{u_1 u_7}, u_7 \mapsto \frac{1}{u_7}, \\ g_3 : u_1 &\mapsto u_7, u_2 \mapsto -u_2, u_3 \mapsto u_3, u_4 \mapsto -u_4, u_5 \mapsto u_6, u_6 \mapsto u_5, u_7 \mapsto u_1. \end{aligned}$$

Define $v_1 = u_1 + u_7$, $v_2 = u_2 u_4$, $v_3 = u_3$, $v_4 = u_4^2$, $v_5 = u_5 + u_6$, $v_6 = u_4(u_5 - u_6)$, $v_7 = u_4(u_1 - u_7)$. Then $k(u_1, u_2, u_3, u_4, u_5, u_6, u_7)^{\langle g_3 \rangle} = k(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and

$$\begin{aligned} g_1 : v_1 &\mapsto \frac{4v_3 v_4 (v_1 v_4 v_5^2 + v_1 v_6^2 + 2v_5 v_6 v_7)}{(v_4 v_5^2 - v_6^2)(v_1^2 v_4 - v_7^2)}, v_2 \mapsto \frac{v_4}{v_2}, v_3 \mapsto v_3, v_4 \mapsto v_4, \\ v_5 &\mapsto \frac{4v_3 v_4 v_5}{v_4 v_5^2 - v_6^2}, v_6 \mapsto \frac{4v_3 v_4 v_6}{v_4 v_5^2 - v_6^2}, v_7 \mapsto -\frac{4v_3 v_4 (2v_1 v_4 v_5 v_6 + v_4 v_5^2 v_7 + v_6^2 v_7)}{(v_4 v_5^2 - v_6^2)(v_1^2 v_4 - v_7^2)}, \\ g_2 : v_1 &\mapsto \frac{4v_1 v_4}{v_1^2 v_4 - v_7^2}, v_2 \mapsto \frac{4v_2 v_3}{v_1^2 v_4 - v_7^2}, v_3 \mapsto \frac{1}{v_3}, v_4 \mapsto \frac{1}{v_4}, v_5 \mapsto -\frac{4v_4 v_5}{v_1^2 v_4 - v_7^2}, v_6 \mapsto \frac{4v_6}{v_1^2 v_4 - v_7^2}, v_7 \mapsto -\frac{4v_7}{v_1^2 v_4 - v_7^2}. \end{aligned}$$

Define

$$\begin{aligned} X_1 &= \frac{1}{v_4}, X_2 = \frac{1}{v_2}, X_3 = \frac{v_1^2 v_4 - v_7^2}{4v_2 v_3}, X_4 = \frac{(v_4 v_5^2 - v_6^2)(v_1^2 v_4 - v_7^2)}{2v_4 v_5 (v_1 v_4 v_5 + v_6 v_7)}, \\ X_5 &= -\frac{(v_1 v_4 v_5^2 + v_1 v_6^2 + 2v_5 v_6 v_7)}{v_1 (v_1 v_4 v_5 + v_6 v_7)}, X_6 = \frac{v_4 (v_1 v_6 + v_5 v_7)}{v_1 v_4 v_5 + v_6 v_7}, X_7 = \frac{v_6}{v_5}. \end{aligned}$$

Then it follows from

$$\begin{aligned} v_1 &= -\frac{2(X_4 - X_1 X_4 X_6 X_7)}{X_1 X_6^2 + X_1 X_7^2 - X_1^2 X_6^2 X_7^2 - 1}, v_2 = \frac{1}{X_2}, v_3 = \frac{X_2 X_4^2}{X_1 X_3 (X_1 X_6^2 - 1)(X_1 X_7^2 - 1)}, v_4 = \frac{1}{X_1}, \\ v_5 &= -\frac{2(X_1 X_4 X_5 X_6 X_7 - X_4 X_5)}{X_1 X_6^2 - X_1 X_6 X_7 + X_1^2 X_6^3 X_7 + X_1 X_7^2 - X_1^2 X_6^2 X_7^2 + X_1^2 X_6 X_7^3 - X_1^3 X_6^3 X_7^3 - 1}, \\ v_6 &= -\frac{2X_4 X_5 X_7 (X_1 X_6 X_7 - 1)}{(X_1 X_6^2 - 1)(X_1 X_6 X_7 - X_1 X_7^2 - X_1^2 X_6^3 X_7 + 1)}, v_7 = \frac{2(X_4 X_6 - X_4 X_7)}{(X_1 X_6^2 - 1)(X_1 X_7^2 - 1)} \end{aligned}$$

that $k(v_1, v_2, v_3, v_4, v_5, v_6, v_7) = k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$. We see that the action of $\langle g_1, g_2 \rangle$ on $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ and that of $\langle \lambda_1, \lambda_2 \rangle$ in Definition 1.30 (ii) are exactly the same. Hence $k(V_{1544})^G$ is rational over $\varphi(L_k^{(3)})$. \square

Proof of Theorem 1.31.

Take a base field k as $k = \mathbb{C}$. The assertion follows from Theorem 1.22 and Theorem 2.1. \square

3. CLASSIFICATION OF GROUPS OF ORDER 128 INTO 115 ISOCLINISM FAMILIES

Φ_j	$G(2^7, i), i \in$	#
1	{1,42,128,159,179,456,837,988,997,1601,2136,2150,2301,2319,2328}	15
2	{5,43,44,106,129,131,153,160,164,180,181,182,457,458,459,476,480,483,498,501, 838,839,840,843,881,894,899,914,989,990,998,999,1000,1001,1002,1003,1004,1602,1603,1604, 1606,1608,1634,1649,1658,1690,1696,2137,2138,2151,2152,2153,2154,2155,2156,2302,2303,2320,2321,2322}	60
3	{8,9,10,11,61,69,99,102,108,111,133,154,206,207,208,294,295,296,307,308, 309,464,466,467,469,490,492,493,496,504,506,507,509,848,884,902,1622,1623,1624,1631, 1639,1640,1641,1646,1668,1669,1670,1671,1685,2306,2307,2308,2309}	53
4	{165,166,167,168,169,183,184,185,186,187,460,461,477,478,481,482,484,485,499,500, 502,503,511,529,549,563,571,574,582,601,648,655,671,686,716,844,845,882,883,895, 896,900,901,915,1009,1010,1011,1012,1013,1014,1015,1016,1017,1018,1019,1020,1021,1022,1023,1024, 1025,1026,1027,1028,1029,1030,1031,1032,1033,1034,1035,1036,1037,1038,1039,1609,1610,1635,1636,1650, 1651,1652,1654,1659,1660,1661,1663,1691,1692,1697,1698,2163,2164,2165,2166,2167,2168,2169,2170,2171, 2172,2173,2174,2175,2176}	105
5	{1005,1006,1007,1008,1605,1607,1704,1714,1720,2139,2157,2158,2159,2160,2161,2162,2304,2305,2323,2324, 2325}	21
6	{209,210,211,297,310,311,312,465,468,470,491,494,495,497,505,508,510,849,885,903,1625,1626,1627, 1628,1632,1642,1643,1647,1672,1673,1674,1675,1676,1677,1686,1687,2310,2311,2312}	39
7	{12,13,14,15,46,54,109,110,132,188,189,190,191,192,193,471,472,473,474,475, 486,487,488,489,846,847,1613,1614,1617,1618,1619}	31
8	{63,64,67,103,104,105,112,113,114,868,869,870,874,888,889,890,892,904,905,906, 910,2140,2141,2142,2143}	25
9	{170,171,172,173,174,175,176,177,178,512,530,550,564,572,575,583,602,649,656,672, 687,717,1116,1117,1118,1119,1120,1121,1122,1123,1124,1125,1126,1127,1128,1129,1130,1131,1132,1133, 1134}	41
10	{1070,1071,1072,1073,1074,1075,1076,1077,1078,1079,1080,1081,1082,1083,1084,1085,1086,1087,1088,1089, 1090,1091,1092,1093,1094,1095,1096,1097,1098,1099,1100,1101,1102,1103,1104,1105,1106,1612,1638,1656, 1657,1665,1666,1667,1694,1695,1700,1701,1702,1703,1706,1709,1711,1716,1718,1722,1723,1726,2194,2195, 2196,2197,2198,2199,2200,2201,2202,2203,2204,2205,2206,2207,2208,2209,2210,2211,2212,2213,2214,2215}	80
11	{1040,1041,1042,1043,1044,1045,1046,1047,1048,1049,1050,1051,1052,1053,1054,1055,1056,1057,1058,1059, 1060,1061,1062,1063,1064,1065,1066,1067,1068,1069,1611,1637,1653,1655,1662,1664,1693,1699,1705,1707, 1708,1713,1715,1717,1721,1724,1725,2177,2178,2179,2180,2181,2182,2183,2184,2185,2186,2187,2188,2189, 2190,2191,2192,2193}	64
12	{6,7,45,107,130,462,463,479,841,842}	10
13	{1107,1108,1109,1110,1111,1112,1113,1114,1115,1710,1712,1719,1727,2257,2258,2259,2260,2261,2262,2263}	20
14	{224,225,226,298,299,300,321,322,323,539,540,542,566,567,569,576,577,578,580,624, 625,628,665,666,669,673,674,677,688,689,690,694,700,701,702,706,722,723,726,1779, 1780,1781,1782,1860,1861,1862,1863,1875,1876,1877,1878,1879,1889,1890,1891,1892,1899,1900,1901,1902}	60
15	{212,213,214,215,216,217,218,219,220,221,222,223,302,303,304,305,306,313,314,315, 316,317,318,319,320,518,519,520,521,534,535,536,537,538,555,556,557,558,584,585, 586,587,594,595,596,597,598,603,604,605,606,607,608,609,610,611,612,650,651,652, 653,654,657,658,659,660,661,679,680,681,682,712,713,714,715,718,719,720,721,1728, 1729,1730,1731,1732,1733,1734,1761,1762,1763,1764,1765,1766,1767,1802,1803,1804,1805,1806,1807,1808, 1817,1818,1819,1820,1821,1822,1823,1832,1833,1834,1835,1836,1837,1838,1839,1840}	116
16	{227,228,229,301,324,325,326,541,543,568,570,579,581,626,627,629,667,668,670,675, 676,678,691,692,693,695,703,704,705,707,724,725,727,1783,1784,1785,1786,1864,1865,1866, 1867,1880,1881,1882,1893,1894,1903,1904}	48
17	{2,3,4,26,27,28,29,30,31,32,33,34,35,47,55,62,70,230,231,232,233,234,235,254,255,256,257,258,270, 271,272,273,274,275,276}	35
18	{1629,1630,1633,1644,1645,1648,1678,1679,1680,1681,1682,1683,1684,1688,1689,2313,2314,2315,2316}	19
19	{871,872,873,875,891,893,907,908,909,911,2144,2145,2146}	13
20	{100,101,115,116,117,886,887}	7
21	{65,66,68,118,119,120,121,876,877,878,879,880}	12
22	{48,49,52,58,60,122,123,124,850,851,852}	11
23	{50,51,53,56,57,59,125,126,127,856,857,858,862,863,864}	15
24	{546,547,548,638,639,640,662,663,664,683,684,685,708,709,710,711,1796,1797,1798,1799}	20
25	{522,523,599,600,621,622,623,1755,1756,1757}	10

Φ_j	$G(2^7, i), i \in$	#
26	{524,525,526,588,589,590,591,592,593,613,614,615,616,617,618,619,620,1746,1747,1748,1749,1750}	22
27	{147,148,149,155,156,157,991,992,993,994}	10
28	{1400,1401,1402,1403,1404,1405,1406,1407,1408,1409,1410,1411,1412,1413,1414,1415,1416,1417,1418,1419,1420,1421,1422,1423,1424,1425,1426,1427,1428,1429,1430,1431,1432,1433,1434,1435,1436,1437,1438,1439,1440,1441,1442,1443,1444,1445,1446,1447,1448,1449,1450,1451,1452,1453,1454,1455,1456,1457,1458,1459,1460,1461,1462,1463,1464,1465,1466,1467,1468,1469,1470,1471,1472,1473,1474,1475,1476,1477,1478,1479,1480,1481,1482,1483,1484,1485,1486,1487,1488,1489,1490,1491,1492,1493,1494,1495,1496,1497,1498,1499,1500,1501,1502,1503,1504,1505,1506,1507,1508,1509,1510,1511,1512,1513,1514,1515,1516,1517,1518,1519,1520,1521,1522,1523,1524,1525,1526,1527,1528,1529,1530,1531,1532,1533,1534,1535,1536,1537,1538,1539,1540,1541,1542,1543}	144
29	{1135,1136,1137,1138,1139,1140,1141,1142,1143,1144,1145,1146,1147,1148,1149,1150,1151,1152,1153,1154,1155,1156,1157,1158,1159,1160,1161,1162,1163,1164,1165,1166,1167,1168,1169,1170,1171,1172,1173,1174,1175,1176,1177,1178,1179,1180,1181,1182,1183,1184,1185,1186,1187,1188,1189,1190,1191,1192,1193,1194,1195,1196,1197,1198,1199,1200,1201,1202,1203,1204,1205,1206,1207,1208,1209,1210,1211,1212,1213,1214,1215,1216,1217,1218,1219,1220,1221,1222,1223,1224,1225,1226,1227,1228,1229,1230,1231,1232,1233,1234,1235,1236,1237,1238,1239,1240,1241,1242,1243,1244,1245,1246,1247,1248,1249,1250,1251,1252,1253,1254,1255,1256,1257,1258,1259,1260,1261,1262,1263,1264,1265,1266,1267,1268,1269,1270,1271,1272,1273,1274,1275,1276,1277,1278,1279,1280,1281,1282,1283,1284,1285,1286,1287,1288,1289,1290,1291,1292,1293,1294,1295,1296,1297,1298,1299,1300,1301,1302,1303,1304,1305,1306,1307,1308,1309,1310,1311,1312,1313,1314,1315,1316,1317,1318,1319,1320,1321,1322,1323,1324,1325,1326,1327,1328,1329,1330,1331,1332,1333,1334,1335,1336,1337,1338,1339,1340,1341,1342,1343,1344}	210
30	{1544,1545,1546,1547,1548,1549,1550,1551,1552,1553,1554,1555,1556,1557,1558,1559,1560,1561,1562,1563,1564,1565,1566,1567,1568,1569,1570,1571,1572,1573,1574,1575,1576,1577}	34
31	{1345,1346,1347,1348,1349,1350,1351,1352,1353,1354,1355,1356,1357,1358,1359,1360,1361,1362,1363,1364,1365,1366,1367,1368,1369,1370,1371,1372,1373,1374,1375,1376,1377,1378,1379,1380,1381,1382,1383,1384,1385,1386,1387,1388,1389,1390,1391,1392,1393,1394,1395,1396,1397,1398,1399}	55
32	{1578,1579,1580,1581,1582,1583,1584,1585,1586,1587,1588,1589,1590,1591,1592,1593,1594,1595,1596,1597,1598,1599,1600}	23
33	{2264,2265,2266,2267,2268,2269,2270,2271,2272,2273,2274,2275,2276,2277,2278,2279,2280,2281,2282,2283,2284,2285,2286,2287,2288,2289,2290,2291,2292,2293,2294,2295,2296,2297,2298,2299,2300}	37
34	{2216,2217,2218,2219,2220,2221,2222,2223,2224,2225,2226,2227,2228,2229,2230,2231,2232,2233,2234,2235,2236,2237,2238,2239,2240,2241,2242,2243,2244,2245,2246,2247,2248,2249,2250,2251,2252,2253,2254,2255,2256}	41
35	{2326,2327}	2
36	{731,732,733,743,744,745,746,747,748,755,756,757,758,765,766,767,768,773,774,775,786,787,788,789,790,791,797,798,799,800,803,804,805,806,807,808,809,815,816,817,818,819,820,824,825,826,827,828,829,831,832,833}	52
37	{242,243,244,245,246,247,265,266,267,268,269,287,288,289,290,291,292,293}	18
38	{236,237,238,239,240,241,248,249,250,251,252,253,259,260,261,262,263,264,277,278,279,280,281,282,283,284,285,286}	28
39	{36,37,38,39,40,41}	6
40	{1996,1997,1998,1999,2000,2001,2002,2003,2004,2005,2006,2007,2008,2009,2010,2038,2039,2040,2041,2042,2043,2044,2045,2046,2047,2048,2049,2050,2051,2052,2053,2054,2055,2056,2057,2058,2059,2060,2061,2062,2063,2064,2065,2075,2076,2077,2078,2079,2080,2081,2082,2083,2084,2085,2086,2087,2088,2089,2098,2099,2100,2101,2102,2103,2104,2105,2106,2107,2108,2109,2116,2117,2118,2119,2120,2121,2122,2129,2130,2131,2132,2133,2134,2135}	84
41	{2011,2012,2013,2014,2015,2016,2017,2018,2019,2026,2027,2028,2029,2030,2031,2032,2033,2034,2035,2036,2037,2066,2067,2068,2069,2070,2071,2072,2073,2074,2090,2091,2092,2093,2094,2095,2096,2097,2110,2111,2112,2113,2114,2115,2123,2124,2125,2126,2127,2128}	50
42	{1930,1931,1932,1933,1934,1935,1936,1952,1953,1954,1955,1956,1957,1973,1974,1975,1976,1989,1990,1991,1992,1993,1994,1995}	24
43	{1924,1925,1926,1927,1928,1929,1945,1946,1947,1948,1949,1950,1951,1966,1967,1968,1969,1970,1971,1972,1983,1984,1985,1986,1987,1988}	26
44	{1918,1919,1920,1921,1922,1923,1937,1938,1939,1940,1941,1942,1943,1944,1958,1959,1960,1961,1962,1963,1964,1965,1977,1978,1979,1980,1981,1982}	28
45	{16,17,18,19,20,21,22,23,24,25}	10

Φ_j	$G(2^7, i), i \in$	#
46	{1740,1741,1742,1743,1744,1745,1773,1774,1775,1776,1777,1778,1813,1814,1815,1816,1828,1829,1830,1831,1850,1851,1852,1853,1854,1855,1856,1857,1858,1859}	30
47	{1735,1736,1737,1738,1739,1768,1769,1770,1771,1772,1809,1810,1811,1812,1824,1825,1826,1827,1841,1842,1843,1844,1845,1846,1847,1848,1849}	27
48	{200,201,202,203,204,205,630,631,632,633,696,697,698,699,728,729,730}	17
49	{1789,1790,1791,1792,1793,1794,1795,1868,1869,1886,1887,1888,1895,1896,1905,1906,1907,1908,1909,1910,1911}	21
50	{544,545,559,565,573,897,898}	7
51	{1787,1788,1870,1871,1872,1873,1874,1883,1884,1885,1897,1898,1912,1913,1914,1915,1916,1917}	18
52	{194,195,196,197,198,199,513,514,515,516,517,531,532,533,551,552,553,554}	18
53	{2317,2318}	2
54	{1615,1616,1620,1621}	4
55	{387,388,389,390,391,392,393,394,395,396}	10
56	{351,352,353,354,355,356,357,358,359,360,361,362,363,364,365,366,367,368,369,370,371,372,373,374,375,376,377,378,379,380,381,382,383,384,385,386}	36
57	{327,328,329,330,331,332,333,334,335,336,337,338,339,340,341,342,343,344,345,346,347,348,349,350}	24
58	{417,418,419,420,421,422,423,424,425,426,427,428,429,430,431,432,433,434,435,436}	20
59	{397,398,399,400,401,402,403,404,405,406,407,408,409,410,411,412,413,414,415,416}	20
60	{446,447,448,449,450,451,452,453,454,455}	10
61	{437,438,439,440,441,442,443,444,445}	9
62	{750,751,752,777,778,779,783,784,795,796,810,811,821,822 }	14
63	{734,735,736,737,738,739,759,760,769,770,771,772,792,793 }	14
64	{763,782,785,812,834,835}	6
65	{753,754,761,762,776,794}	6
66	{742,749,823,830}	4
67	{2020,2021}	2
68	{2022,2023,2024,2025}	4
69	{1751,1752,1753,1754}	4
70	{634,635,636,637}	4
71	{645,646,647}	3
72	{1758,1759,1760}	3
73	{641,642,643,644}	4
74	{1800,1801}	2
75	{527,528}	2
76	{560,561,562}	3
77	{75,76,77,78,83,84,85,86}	8
78	{79,80,81,82,92,93,94,95,96,97}	10
79	{916,917,918,919,920,921,938,939,940,941,942,943,956,957,958,959,960,961,964,965,966,967,968,969}	24
80	{950,951,952,975,976,977,982,983,987}	9
81	{947,948,949,972,973,974,978,979,980,981,984,985,986}	13
82	{912,913}	2
83	{853,854,855}	3
84	{859,860,861,865,866,867}	6
85	{2147,2148,2149}	3
86	{780,781}	2
87	{836}	1
88	{801,802}	2
89	{740,741}	2
90	{813,814}	2
91	{764}	1
92	{934,935}	2
93	{936,937}	2
94	{931,932,933}	3
95	{928,929,930}	3
96	{71,72,73,74}	4
97	{98}	1

Φ_j	$G(2^7, i), i \in$	#
98	{89,90,91}	3
99	{87,88}	2
100	{924,925,926,927}	4
101	{922,923}	2
102	{970,971}	2
103	{944,945,946}	3
104	{962,963}	2
105	{953,954,955}	3
106	{144,145}	2
107	{140,141,142,143}	4
108	{150,151,152}	3
109	{995,996}	2
110	{158}	1
111	{134,135}	2
112	{136,137}	2
113	{161,162,163}	3
114	{138,139}	2
115	{146}	1
		2328

Table 2 Classification of groups of order 128 into 115 isoclinism families

Remark 3.1. For nine groups $G = G(2^6, i)$ which belong to Φ_{16} with $B_0(G) \neq 0$, we have

$$\begin{aligned}
&G(2^6, 149) \times C_2 \simeq G(2^7, 1783), \quad G(2^6, 150) \times C_2 \simeq G(2^7, 1784), \quad G(2^6, 151) \times C_2 \simeq G(2^7, 1785), \\
&G(2^6, 170) \times C_2 \simeq G(2^7, 1864), \quad G(2^6, 171) \times C_2 \simeq G(2^7, 1865), \quad G(2^6, 172) \times C_2 \simeq G(2^7, 1866), \\
&G(2^6, 177) \times C_2 \simeq G(2^7, 1880), \quad G(2^6, 178) \times C_2 \simeq G(2^7, 1881), \quad G(2^6, 182) \times C_2 \simeq G(2^7, 1893).
\end{aligned}$$

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